Closure correlations in paving and low-corank matroids

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Abstract

Sample an independent set after deleting two marked elements of a finite matroid, with probability proportional to the product of its element weights. We prove that the two events that the sampled set spans the marked elements are positively correlated for every paving matroid and for every matroid of corank at most three. In both cases the covariance numerator has nonnegative coefficients in the individual weights. The paving result follows from a monomial-preserving circuit exchange and extends to a condition on circuits through just one mark. The low-corank result uses complementary spanning sets in the dual and a coefficient inequality for two pencils of lines. Exact minor and series and parallel extension identities give further closure properties, and the proofs yield strictness criteria for positive weights. We also give a portable integer certificate for the binary rank-four consequence of earlier reported computations, including its forest-connectivity specialization.

Keywords. Matroid; closure correlation; paving matroid; independent set; forest connectivity.
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1 Introduction and main results

Let M be a finite matroid on E, with distinct marked elements e, f. Put $X = E \setminus \{e, f\}$. For nonnegative finite weights $\mathbf{x} = (x_g)_{g \in X}$, sample an independent set A of $M|X$ according to

$$\mathbb{P}_{\mathbf{x}}(A) = \frac{x^A}{Z(\mathbf{x})}, \qquad x^A = \prod_{g \in A} x_g, \qquad Z(\mathbf{x}) = \sum_{A \in \mathcal{I}(M|X)} x^A.$$

The empty set gives $Z \geq 1$. We study whether spanning one marked element makes spanning the other more likely. This is the closure-correlation question of Çiçeksiz and Ravichandran [1, Conjecture 7.1]. For a graphic matroid, a marked edge is in $\mathrm{cl}_M(A)$ exactly when its endpoints are connected by the sampled forest.

For $i, j \in \{0, 1\}$ define the category polynomials

$$I_{ij}(\mathbf{x}) = \sum_{\substack{A \in \mathcal{I}(M|X) \\ \mathbf{1}_{\{e \in \mathrm{cl}_M(A)\}} = i,\ \mathbf{1}_{\{f \in \mathrm{cl}_M(A)\}} = j}} x^A, \qquad \Gamma_{M;e,f} = I_{11} I_{00} - I_{10} I_{01}. \tag{1}$$

Then

$$\mathrm{Cov}_{\mathbf{x}}\big(\mathbf{1}_{\{e \in \mathrm{cl}_M(A)\}}, \mathbf{1}_{\{f \in \mathrm{cl}_M(A)\}}\big) = \frac{\Gamma_{M;e,f}(\mathbf{x})}{Z(\mathbf{x})^2}. \tag{2}$$

For $D \subseteq X$ we also write $x_D = \prod_{g \in D} x_g$. Write $P \succeq_{\mathrm{coef}} 0$ when every coefficient of P is nonnegative. The stronger coefficientwise target was proposed and tested computationally in [1, Section 7].

Recall that a rank-r matroid is *paving* if every circuit has at least r elements. We use the standard rank, closure, circuit, duality and simplification conventions of [3]; the corank is $|E| - \operatorname{rk} M$.

Theorem 1.1 (A circuit condition at one mark)**.** *Suppose every circuit of M containing e and avoiding f has size at least* $\operatorname{rk} M$*. Then $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$. The same conclusion holds with e and f interchanged.*

Theorem 1.2 (Corank at most three)**.** *If $|E| - \operatorname{rk} M \leq 3$, then $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$ for every pair of distinct marked elements.*

Theorem 1.1 applies to every paving matroid. The parallel identities proved below extend this to every matroid with paving simplification. In particular it applies to all matroids of rank at most three. Theorem 1.2 covers a complementary range: its dual argument transports the probability law to spanning sets and the events to coloop events.

The reported coefficient test for $\mathrm{PG}(3,2)$ in [1, Section 7] implies positivity for every binary matroid of rank at most four by restriction and parallel extension. Appendix A gives that deduction together with a self-contained independent certificate on the $\binom{13}{6} = 1716$ eight-point restrictions with two fixed marks.

Relation to earlier correlation inequalities. The ordinary edge-Rayleigh difference compares inclusion of marked elements in a sampled independent set. The present events instead ask whether a set avoiding the marks spans them. This distinction can be seen algebraically. Put $a = I_{00}, b = I_{10}, c = I_{01}, d = I_{11}$ and $Z = a + b + c + d$. Let j generate the category-00 sets for which adjoining e puts f in closure. The full independent-set polynomial, with marked weights s, t, is

$$F(s,t) = Z + s(a+c) + t(a+b) + st(a-j).$$

Its marked-edge Rayleigh difference is

$$F_s F_t - F F_{st} = (a+c)(a+b) - Z(a-j) = Zj - \Gamma_{M;e,f}. \tag{3}$$

Erickson [2, Lemma 4.0.5 and Theorem 4.4.1] develops paving exchange counts and proves a coefficientwise lower comparison of the independent-set and basis Rayleigh differences. Equation (3) shows that closure positivity requires an upper bound $F_s F_t - F F_{st} \preceq_{\mathrm{coef}} Zj$, whereas a lower Rayleigh bound gives an upper bound on $\Gamma_{M;e,f}$. We therefore construct a monomial-preserving injection between the closure categories using classical fundamental circuits. The corank proof uses a separate two-pencil counting argument. Minor and coefficient reduction, also present in [2], organize the finite binary consequence.

2 A recoverable circuit exchange

Proof of Theorem 1.1. If a mark is a loop, its indicator is constant and $\Gamma = 0$. If either exclusive category is empty, $I_{10} I_{01} = 0$ and the conclusion is immediate. Otherwise let $r = \operatorname{rk} M$, fix a total order on X, and take

$$A \in \mathcal{I}(M|X), \quad e \in \operatorname{cl}(A),\ f \notin \operatorname{cl}(A), \qquad B \in \mathcal{I}(M|X), \quad f \in \operatorname{cl}(B),\ e \notin \operatorname{cl}(B).$$

We construct an injection from these ordered pairs to category 11 times category 00, preserving the multiset union and hence every monomial.

The independent set A spans e, so $A \cup \{e\}$ contains a unique circuit through e. By hypothesis its size is at least r. Since A does not span f, it is nonspanning, and therefore $|A| \leq r-1$. It follows that

$$|A| = r-1, \qquad A \cup \{e\} \text{ is a circuit.}$$

In particular $\mathrm{cl}(A)$ is a hyperplane.

Let K_f be the unique circuit in $B \cup \{f\}$. There exists $v \in K_f \setminus \{f\}$ outside $\mathrm{cl}(A)$: otherwise $f \in \mathrm{cl}(K_f \setminus \{f\}) \subseteq \mathrm{cl}(A)$, a contradiction. Choose the least such v and set

$$C = A \cup \{v\}, \qquad D = B \setminus \{v\}.$$

The set C is an independent r-set, hence a basis and a category-11 set. The subset D of B is independent and does not span e. It does not span f either: a circuit in $D \cup \{f\}$ would be a second circuit in $B \cup \{f\}$ avoiding v, whereas its unique circuit is K_f. Thus D belongs to category 00.

Since $v \notin \mathrm{cl}(A)$, it is not in A, and $x^A x^B = x^C x^D$, including any squared variables from $A \cap B$. To recover the input, take the unique fundamental e-circuit in $C \cup \{e\}$. It equals $A \cup \{e\}$, so it determines A, then the singleton $C \setminus A = \{v\}$, and finally $B = D \cup \{v\}$. This recovery proves injectivity, and hence the coefficientwise inequality. □

The hypothesis concerns only circuits through one mark that avoid the other. Thus short circuits elsewhere in M, and short circuits through both marks, are permitted. The exchange uses the fundamental f-circuit of B and allows $|B| < r-1$.

Corollary 2.1 (Strictness for paving matroids)**.** *Let M be paving of rank $r \geq 3$, with strictly positive unmarked weights. Then the covariance in* (2) *is positive if and only if $E \setminus \{e, f\}$ spans M.*

Proof. If the deletion does not span, at least one mark lies outside its closure, so that mark's indicator is constantly false. If it spans, choose a basis B in the deletion. Its degree-r monomial has coefficient at least one in $I_{11}I_{00}$, using the empty set in the second factor. Each nonzero exclusive category polynomial has degree $r-1$, by the circuit-size argument above. Thus $I_{10}I_{01}$ has degree $2r-2 > r$ and cannot cancel that coefficient. All other coefficients of Γ are nonnegative. □

3 Parallel extensions and coefficient locality

A marked loop gives $\Gamma = 0$. Parallel nonloop marks have identical closure indicators, so $I_{10} = I_{01} = 0$. Unmarked loops have no effect on the independent-set polynomials.

Lemma 3.1 (Parallel extensions)**.** *Coefficientwise closure positivity is preserved by adding a parallel copy of any marked or unmarked element.*

Proof. A copy of an unmarked nonloop g, with weight t, replaces x_g by $x_g + t$ in each category polynomial and in Γ. This preserves nonnegative coefficients.

For a new unmarked copy of e, use a, b, c, d as above. Split $a = j + k$, where j counts category-00 sets for which adjoining e puts f in closure. A set can accept the new copy precisely when it does not already span e. The new categories are consequently

$$a' = a, \qquad c' = c, \qquad b' = b + tk, \qquad d' = d + t(c+j).$$

Hence

$$\Gamma' = \Gamma + t\big((c+j)a - kc\big) = \Gamma + tj(a+c) \succeq_{\mathrm{coef}} \Gamma. \tag{4}$$

The case of a copy of f is symmetric. Loop copies are loops. □

Corollary 3.2. *If the simplification of M is paving, then $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$ for every marked pair. In particular the assertion holds when* $\operatorname{rk} M \leq 3$.

Proof. For nonparallel nonloop marks choose a simplification retaining both marks. Apply Theorem 1.1 and restore parallel classes by Lemma 3.1, followed by loops. The other marked cases were handled above. Every simple rank-three matroid is paving; the same conclusion in smaller ranks follows from the loop and parallel cases or directly from Theorem 1.1. □

Lemma 3.3 (Coefficient locality)**.** *Let $I, T \subseteq X$ be disjoint. If I is dependent, then $[x_I^2 x_T]\Gamma_{M;e,f} = 0$. If I is independent, then*

$$[x_I^2 x_T]\Gamma_{M;e,f} = [x_T]\Gamma_{(M/I)|(T\cup\{e,f\});e,f}. \tag{5}$$

Consequently coefficientwise closure positivity is preserved by deletion and contraction of unmarked elements. Any negative coefficient of a rank-r marked matroid has a marked minor witness on at most $2r$ elements.

Proof. Exponent two forces every element of I into both independent factors. This is impossible when I is dependent. Otherwise the two factors are $I \cup U, I \cup V$, where (U, V) is an ordered partition of T. Independence passes to M/I, and $h \in \mathrm{cl}_M(I \cup U)$ is equivalent to $h \in \mathrm{cl}_{M/I}(U)$ for $h = e, f$. Deleting the remaining unmarked elements preserves these ranks and closure tests. This proves (5) with each ordered factorization counted once.

Deletion is the constant coefficient in a variable. Contraction of a nonloop g is its squared coefficient. Contraction of a loop equals deletion, so positivity is preserved in that case as well.

If a coefficient is negative, a contributing pair from $I_{10}I_{01}$ exists. Both factors are independent and nonspanning, so each has size at most $r - 1$. Thus $2|I| + |T| \leq 2r - 2$. The witness in (5) has $|T| + 2 \leq 2r - 2|I| \leq 2r$ elements. □

Lemma 3.4 (Direct sums)**.** *If the marks belong to different components of M, then $\Gamma = 0$. If both belong to a component M_0, then $\Gamma_M = \Gamma_{M_0} Z^2_{M\setminus E(M_0)}$.*

Proof. Independent sets and their weights factor over components, and closure of a marked element depends only on its component. The assertions follow by factoring the category polynomials. □

Proposition 3.5 (Series extensions)**.** *The all-pair coefficientwise closure-positivity property is preserved by a series extension.*

Proof. Let M' replace an element g by the series pair g, h, using the dual of parallel extension. Circuits avoiding g are unchanged; a circuit containing g replaces it by $\{g, h\}$. This description also covers an old loop, while extension at a coloop gives two coloops.

First take marks outside the series pair. Write each old category as $I_{ij} = a_{ij} + zb_{ij}$, where $z = x_g$, and write $\Gamma_M(z) = G_0 + zG_1 + z^2G_2$. With $s = x_g, t = x_h$, selecting neither or one of the series elements has the independence and marked closure statuses of its old part avoiding g; selecting both has those of the old part with g adjoined. Hence

$$I'_{ij} = (1 + s + t)a_{ij} + stb_{ij}, \qquad \Gamma_{M'} = (1 + s + t)^2G_0 + st(1 + s + t)G_1 + s^2t^2G_2.$$

All prefactors have nonnegative coefficients.

If $g = e$ is marked and f is another old mark, put $(a, b, c, d) = (I_{00}, I_{10}, I_{01}, I_{11})$ and $t = x_h$. Without h, the mark e cannot be closed, so the old categories collapse according to their f-status. With h, every old independent set avoiding the marks can accept it, and both old statuses are restored. Thus

$$a' = (1 + t)a + b, \quad b' = tb, \quad c' = (1 + t)c + d, \quad d' = td, \qquad \Gamma_{M';e,f} = t(1 + t)\Gamma_{M;e,f}.$$

The case with h marked is symmetric. If both series elements are marked, a set avoiding both cannot close either and $\Gamma = 0$. These cases exhaust the marked pairs. □

Together the identities prove positivity for every minor of a matroid obtained by series and parallel extensions and direct sums from the all-pair-positive seed classes established here. This enlarges the literal seed classes. For example, extend one element of $U_{4,8}$ twice in series. The resulting simple rank-six, ten-element matroid has corank four and retains five-element circuits on the seven untouched elements. It is therefore nonpaving, while Proposition 3.5 proves all-pair closure positivity. Contracting the two new elements recovers its nonbinary minor $U_{4,8}$.

4 Two pencils in the dual

We first isolate the coefficient count needed for the dual experiment. Let G be a finite simple bipartite graph with vertex classes L, H and edge variables w_g. Define

$$Z_G = \prod_{g \in E(G)} (1 + w_g) - 1, \qquad K_G = \sum_g w_g,$$
$$R_G = \sum_{u \in L} \left(\prod_{g \ni u} (1 + w_g) - 1 \right), \qquad C_G = \sum_{v \in H} \left(\prod_{g \ni v} (1 + w_g) - 1 \right).$$

Thus R_G and C_G generate nonempty subsets of individual row-stars and column-stars, while K_G generates single edges.

Lemma 4.1 (Two-star concentration)**.** *For every such graph, $K_G Z_G - R_G C_G \succeq_{\mathrm{coef}} 0$.*

Proof. Each factor is multiaffine. A term of $R_G C_G$ containing w_g^2 has its two factors on the row and column of g. Their only common edge is g, so every other variable has a forced factor. Its coefficient is at most one, and $K_G Z_G$ supplies one. Two different squared variables cannot occur in either product.

For squarefree support S of size $k \geq 2$, the coefficient in $K_G Z_G$ is k. The other product counts ordered nonempty partitions $S = A \sqcup B$ with A in a row and B in a column. For $k = 2$ there are two. For $k \geq 3$, if S is a row-star, the column factor is a singleton, giving k partitions; the column-star case is symmetric. Otherwise S contains disjoint edges $(u, w), (z, v)$. A row and a column covering both edges must be either (u, v) or (z, w). The edges covered by both of these cross-pairs are exactly those two disjoint edges. A third edge therefore rules out one of the pairs. For the remaining pair every edge is forced except its possible intersection edge, giving at most two partitions, and hence at most k. Constant and squarefree linear coefficients are zero in both products. These cases cover all monomials. □

Proof of Theorem 1.2. Let $N = M^*$ have rank $c \leq 3$. For $A \subseteq X$ put $B = X \setminus A$ and $S = B \cup \{e, f\}$. The dual rank formula gives

$$A \in \mathcal{I}(M) \iff S \text{ spans } N, \qquad h \in \mathrm{cl}_M(A) \iff h \text{ is a coloop of } N|S \tag{6}$$

for $h = e, f$ when S spans. Let $J_{ij}(\mathbf{y})$ generate these spanning complements with the indicated coloop statuses. Then

$$I_{ij}(\mathbf{x}) = x_X J_{ij}(\mathbf{x}^{-1}), \qquad \Gamma_M(\mathbf{x}) = x_X^2 (J_{11} J_{00} - J_{10} J_{01})(\mathbf{x}^{-1}).$$

All exponents in $J_{11}J_{00} - J_{10}J_{01}$ lie in $\{0, 1, 2\}$. Multiplication by x_X^2 reverses each exponent k to $2 - k$, so coefficientwise positivity is preserved. This is a polynomial identity; no reciprocal weight needs to be evaluated when a weight is zero.

If a mark is a loop of N, its coloop event is constantly false. If the marks are parallel nonloops of N, neither is a coloop of $N|S$. Thus we may assume they are independent, and $c \geq 2$. For $c = 2$ every B is admissible. Partition the unmarked nonloops of N into P parallel to e, Q parallel to f, and the remaining elements T, and denote the loops by W. In the original A variables write $p = x_P, q = x_Q, t = x_T$ and $U_D = \prod_{g \in D}(1 + x_g)$. The closure events require respectively $P \cup T$ and $Q \cup T$ to be contained in A. Direct multiplication gives

$$\Gamma_M = pqt\, U_P U_Q (U_T - t) U_W^2 \succeq_{\text{coef}} 0.$$

It remains to consider $c = 3$. Put $L_0 = \text{cl}_N\{e, f\}$. Each parallel class outside L_0 belongs to a unique line through e and a unique line through f. These two pencils define a simple bipartite graph G, with one edge for each outside parallel class. Indeed, two nonparallel outside classes on both the same e-line and the same f-line would force those lines to coincide and contain e, f and an outside element, contrary to the definition of L_0. For an outside class D_g substitute

$$w_g = \prod_{h \in D_g} (1 + y_h) - 1.$$

Its coefficients are nonnegative and it generates nonempty subsets of that class. Use Z_G, R_G, C_G, K_G after this substitution, with rows indexed by the f-pencil and columns by the e-pencil.

Partition the unmarked elements in L_0 into P parallel to e, Q parallel to f, the other nonloops T, and the loops W. Now put $U_D = \prod_{h \in D}(1 + y_h)$. Admissibility of B means choosing a nonempty outside subset, since $\{e, f\}$ has rank two. Its total polynomial is

$$J = Z_G U_P U_Q U_T U_W.$$

The event that e is a coloop means that $B \cup \{f\}$ has rank two. The outside subset must lie on one f-line; among nonloops of L_0, only those parallel to f may be selected. Thus its polynomial is $R_G U_Q U_W$. Similarly the other event has polynomial $C_G U_P U_W$. Both events occur precisely when all selected nonloops belong to one outside parallel class: the two distinct pencil lines have rank-one intersection. Consequently $J_{11} = K_G U_W$ and

$$\begin{aligned} J_{11}J_{00} - J_{10}J_{01} &= J_{11}J - (J_{11} + J_{10})(J_{11} + J_{01}) \\ &= U_P U_Q U_W^2 (K_G Z_G U_T - R_G C_G) \succeq_{\text{coef}} 0. \end{aligned}$$

For the last step, write the bracket as $(K_G Z_G - R_G C_G) + K_G Z_G (U_T - 1)$ and apply Lemma 4.1 and the nonnegative substitutions above. Exponent reversal proves the result for M. □

Corollary 4.2 (Low-corank strictness)**.** *Assume all weights are strictly positive and put $N = M^*$. If the marks are dependent in N, then $\Gamma_M = 0$. For independent marks in rank-two N, the covariance is positive precisely when N has an unmarked nonloop parallel to neither mark. For independent marks in rank-three N, use the outside-class graph G and the set T constructed in the preceding proof. The covariance is positive precisely when $T \neq \varnothing$ or the edges of G do not all share a single vertex.*

Proof. The dependent-mark and rank-two assertions follow from the displayed factorizations above. In rank three, G has an edge. If all its edges share a row, $R_G = Z_G, C_G = K_G$; the column case is

symmetric. Thus $K_G Z_G - R_G C_G = 0$. Otherwise G contains disjoint edges g, h. The coefficient of $w_g^2 w_h$ in $K_G Z_G$ is one, while in $R_G C_G$ it is zero: the squared edge forces its row and column, neither containing h. Thus the nonnegative difference is nonzero. Also $K_G Z_G (U_T - 1)$ is nonzero exactly when T is nonempty. Positive parallel-class substitution and exponent reversal preserve strict positivity of these nonnegative polynomials. The covariance denominator is positive throughout. □

A The binary certificate and forest connectivity

Corollary A.1 (Binary consequence of the reported geometry test)**.** *Every finite binary matroid of rank at most four satisfies* $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$ *for every pair of distinct marked elements.*

The coefficient test for $\mathrm{PG}(3,2)$ was reported in [1, Section 7]. Every simple binary matroid of rank at most four is a restriction of that geometry, whose distinct marked pairs are equivalent under ambient linear changes of basis. Setting deleted weights to zero and using Lemma 3.1 gives the stated consequence. The following independent certificate supplies a smaller exhaustive check using the coefficient-locality reduction.

Independent finite certificate for Corollary A.1. Ranks at most three are covered by Corollary 3.2. By Lemma 3.3, a negative coefficient in rank four would give a marked binary minor on at most eight elements. A nonempty squared support contracts to rank at most three and cannot be negative. By Lemma 3.1, negativity would persist in a simplification retaining the two nonparallel nonloop marks. Thus a simple binary rank-four witness with at most eight elements would exist.

Represent its columns by distinct nonzero vectors of $\mathbb{F}_2^4$. A linear change of basis sends the marked vectors to $1 = (1,0,0,0)$ and $2 = (0,1,0,0)$. Extend the column set to eight distinct nonzero vectors, if necessary. The original coefficient is a constant coefficient in the added variables, so is preserved by this extension. The remaining six columns are one of the $\binom{13}{6}$ subsets of the other thirteen points. Every resulting eight-point configuration has rank four, because a rank-three binary subspace has only seven nonzero points.

The accompanying `verify.py` enumerates exactly these subsets, computes every rank by binary elimination, forms (1), and multiplies the category polynomials with integer arithmetic. It checks every coefficient, including proper supports and repeated variables. Table 1 records its full-support distribution; all 274380 nonzero coefficients checked in total are positive. This exhaustive calculation excludes the witness above and proves the corollary. □

Full-support coefficient	0	2	3	4	5	6	8	9	10	12
Configurations	36	48	192	48	48	96	436	480	312	20

Table 1: The coefficient of the product of all six unmarked variables in the 1716 binary configurations. The verifier also checks every other coefficient.

For comparison, the preceding results localize any rank-four failure to an especially concrete support.

Corollary A.2. *If a rank-four matroid has a negative closure coefficient, it has a simple nonbinary rank-four marked minor on exactly eight elements with a negative full squarefree coefficient. Each mark lies on a triangle avoiding the other.*

Proof. Use locality and simplification as above. A witness on at most seven elements has corank at most three, contradicting Theorem 1.2. Its squared support is empty, so the witness has full

squarefree support. Binary witnesses are excluded by Corollary A.1. In a simple rank-four matroid the failure of the circuit condition at a mark requires a circuit of size three through it avoiding the other. □

Corollary A.3 (Five-vertex forest connectivity)**.** *Let G be a finite graph on at most five vertices, with arbitrary nonnegative finite edge weights. Sample an acyclic edge set with mass proportional to its edge-weight product. Any two endpoint-connectivity events are positively correlated.*

Proof. Add marked edges joining the two endpoint pairs, allowing parallel marked copies of existing edges. The graphic matroid is binary and has rank at most four. Its independent sets after deleting the marks are exactly the original forests. Closure of a marked edge is connectivity of its endpoints. Apply Corollary A.1 and (2). Equal endpoints give a constant event and require no additional argument. □

Reproducibility and declarations

The source package contains the complete integer verifier and its expected output. It uses only the Python standard library. The rank and category reduction proved in Appendix A specifies its exhaustive scope.

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