Chollet's inequality for positive semidefinite matrices of rank at most four

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Luminary Oddities

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Abstract

We prove quantitative self-conjugate Chollet inequalities for all complex Hermitian positive semidefinite matrices of rank at most four in orders eight and nine, with respective factors $243/320$ and $1215/1352$. A short extension of the occupation-border comparison of Li, Zhao and Luo balances its two multiplicative losses. The input is the previously established singular order-five factor $3/5$; no new certificate is needed. Together with Rodtes' order-four theorem and Ma's theorem for orders at least ten, the new estimates imply Chollet's inequality in every order when both factors have rank at most four. The coefficient-one inequality is strict for positive diagonal in orders at least five.

Keywords: permanent; positive semidefinite matrix; Hadamard product; Chollet inequality; rank four; border comparison.

1 Statements and relation to earlier work

For a matrix $A = (a_{ij})$ of order n, put

$$\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} a_{i,\sigma(i)}, \qquad P(A) = \operatorname{per} A, \qquad Q(A) = \operatorname{per}(A \circ \overline{A}).$$

Here $\circ$ denotes entrywise multiplication and $\overline{A}$ entrywise conjugation. All positive semidefinite matrices in this paper are complex Hermitian. Chollet's conjecture asks whether $\operatorname{per}(A \circ B) \le (\operatorname{per} A)(\operatorname{per} B)$ for all such pairs [1].

Theorem 1.1. *Let $A \succeq 0$ have rank at most four and order $n \in \{5, 6, 7, 8, 9\}$. Then*

$$Q(A) \le c_n P(A)^2, \qquad c_5 = c_6 = \frac{3}{5}, \quad c_7 = \frac{27}{40}, \quad c_8 = \frac{243}{320}, \quad c_9 = \frac{1215}{1352}.$$

Each inequality is strict if all diagonal entries are positive. If a diagonal entry is zero, $P(A) = Q(A) = 0$.

The paired conclusion below restricts both factors to rank at most four. The proof spends a strict reserve from the singular-five base by balancing two border losses; the large-order theorem is a separately credited input.

The order-five bound and its exact certificate are from the author's earlier order-seven paper [2]. The new full-domain cases are orders eight and nine; the factor in order seven improves the previous $192/245$. No tight-frame, spectral, row-cap or real-entry condition is imposed.

Corollary 1.2 (All orders, rank at most four)**.** *For every $n \ge 1$ and every pair $A, B \succeq 0$ of order n with* $\operatorname{rank} A \le 4$ *and* $\operatorname{rank} B \le 4$,

$$\operatorname{per}(A \circ B) \le (\operatorname{per} A)(\operatorname{per} B).$$

For $5 \le n \le 9$, the stronger coefficient c_n in Theorem 1.1 applies. For $n \ge 10$, Ma's coefficient $\rho_4 = 999991742359/10^{12} < 1$ applies. The coefficient-one inequality is strict for $n \ge 5$ if both diagonals are positive. If either matrix has a zero diagonal entry, both sides vanish.

Corollary 1.2 is a synthesis: Rodtes proves the complex order-four case [3], and Ma proves the rank-at-most-four case for every order at least ten [4, Theorem 1.2]. Ma also treats tight order-nine correlation matrices, but that separate result is not used here. Theorem 1.1 supplies the missing unrestricted rank-four small orders.

The identities and occupation positivity used below come from Li–Zhao–Luo [5, Proposition 3.2 and Eq. (4.1)]. Their loss-free comparison is insufficient for continuation beyond its stated range. We instead allow both coefficient comparisons to have losses and use the existing quantitative margin to pay for them. This is an elementary refinement of that mechanism. The singular-five certificate and its principal-minor sum-of-squares construction are reused from [2]; their feature framework follows Lavi [6]. The paper uses the cited positivity framework and established certificate.

2 Positivity and the reused base

If $a_{ii} = 0$ in a PSD matrix, the two-by-two principal minors give $|a_{ij}|^2 \le a_{ii}a_{jj} = 0$. Its ith row and column vanish, hence $P(A) = Q(A) = 0$. Otherwise positive diagonal congruence normalizes A to a correlation matrix. For $D = \operatorname{diag}(d_i)$, $d_i > 0$,

$$P(DAD) = \Big(\prod_i d_i^2\Big)P(A), \qquad Q(DAD) = \Big(\prod_i d_i^4\Big)Q(A). \tag{1}$$

Rank is unchanged. Thus normalization preserves all self-conjugate bounds considered here without division by $P(A)$.

We recall the occupation positivity needed for the border calculation. For PSD matrices $M_1, \dots, M_\ell$ of order m, $\operatorname{per}(\sum_j t_j M_j)$ has nonnegative real coefficients. To see this, realize their Gram vectors $u_{i,j}$ in mutually orthogonal Hilbert spaces and set $v_i = \bigoplus_j \sqrt{t_j}u_{i,j}$ for $t_j \ge 0$. The unnormalized symmetric tensor

$$T(v) = \sum_{\sigma \in \mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \quad \text{satisfies} \quad \|T(v)\|^2 = m!\ \operatorname{per}\Big(\sum_j t_j M_j\Big).$$

Group the complete tensor sums by occupation counts in the orthogonal summands. Different profiles are orthogonal, so their squared norms, divided by $m!$, are the polynomial coefficients. This argument covers singular matrices.

We use the following prior computer-assisted theorem as the base.

Proposition 2.1 (Singular-five estimate [2])**.** *If $M \ge 0$ has order five and $\det M = 0$, then $Q(M) \le (3/5)P(M)^2$, strictly when its diagonal is positive.*

For clarity, its certificate proves the exact polynomial identity

$$D(3P^2 - 5Q) = \frac{1}{240} \sum_{g \in S_5 \times \{1,\text{transpose}\}} g\Big(\sum_{\nu=1}^{38} w_\nu b_\nu^* N_\nu b_\nu\Big) + \det(M)h(M). \tag{2}$$

Here $D = 260963607394941665280 > 0$, each w_ν is a product of principal minors, and every N_ν is positive definite. The formal transpose involution becomes conjugation on Hermitian matrices. The exact data, interpretation and proof are those of [2]. On the singular PSD domain, the last term vanishes and all other terms are nonnegative.

Strictness has a useful concrete witness. One factor-free positive definite block contains all 119 non-identity permutation monomials. A singular matrix with positive diagonal cannot be diagonal. A nonzero off-diagonal entry gives a positive transposition monomial, so this block is nonzero. The accompanying unchanged verifier reconstructs all 6,210 coefficient identities and checks the 38 positive-definiteness witnesses. These exact data are included solely to make the reused base inspectable, not as a new contribution.

3 A controlled-loss border comparison

The following extension records explicitly how to spend a quantitative input margin in the border identities of [5].

Lemma 3.1. *Let $A \geq 0$ be a correlation matrix of order $m+1$, and write*

$$A = \begin{pmatrix} B & x \\ x^* & 1 \end{pmatrix}, \qquad H = B - xx^*, \qquad R_t = H + t\,xx^*, \qquad t \geq 1.$$

Define

$$b_m(t) = \max_{0 \leq k \leq m} \frac{t^k}{k+1}, \qquad d_m(t) = \max_{0 \leq s \leq m} \frac{s+1}{t^{2s}}.$$

Then $R_t \geq 0$, $\operatorname{rank} R_t \leq \operatorname{rank} A$*, its diagonal is positive, and*

$$P(R_t) \leq b_m(t) P(A), \qquad Q(A) \leq d_m(t) Q(R_t). \tag{3}$$

Consequently, an order-m rank-at-most-four bound $Q \leq c_m P^2$ implies

$$Q(A) \leq c_m b_m(t)^2 d_m(t) P(A)^2 \qquad (\operatorname{rank} A \leq 4). \tag{4}$$

Strictness for positive diagonal propagates from the order-m bound.

Proof. Schur congruence gives $H \geq 0$ and $\operatorname{rank} H = \operatorname{rank} A - 1$. Thus $\operatorname{rank} R_t \leq \operatorname{rank} A$ and $(R_t)_{ii} = 1 + (t-1)|x_i|^2 \geq 1$.

Set $z_i = |x_i|^2$ and

$$U = H \circ \overline{H}, \quad V = H \circ \overline{xx^*} + \overline{H} \circ xx^*, \quad W = zz^T.$$

These are PSD: U follows from Schur positivity, each summand of V is a diagonal congruence of H or $\overline{H}$, and W has rank at most one. Expand

$$F(t) = P(H + t\,xx^*) = \sum_{k=0}^{m} \alpha_k t^k, \qquad \Gamma(a,b) = P(U + aV + bW) = \sum_{\substack{r,s \geq 0 \\ r+s \leq m}} \gamma_{rs} a^r b^s.$$

Occupation positivity gives $\alpha_k, \gamma_{rs} \geq 0$.

For $B(i \mid j)$ obtained by deleting row i and column j, border expansion yields

$$P(A) = P(B) + \sum_{i,j} x_i \overline{x_j} \operatorname{per} B(i \mid j) = F(1) + F'(1) = \sum_k (k+1)\alpha_k.$$

In particular the cofactor orientation is unchanged for complex x. The matrix $A \circ \overline{A}$ has upper block $U + V + W$ and border z. The same expansion in direction W gives

$$Q(A) = \Gamma(1,1) + \partial_b \Gamma(1,1) = \sum_{r+s \leq m} (s+1)\gamma_{rs}, \tag{5}$$

$$Q(R_t) = \Gamma(t, t^2) = \sum_{r+s \leq m} t^{r+2s} \gamma_{rs}. \tag{6}$$

The first comparison in (3) follows coefficientwise from $t^k \leq b_m(t)(k+1)$. For the second, $t \geq 1$ ensures

$$s + 1 \leq d_m(t) t^{2s} \leq d_m(t) t^{r+2s}.$$

Apply this to (5)–(6). Normalize R_t using (1), apply its order-m bound, and use nonnegativity of PSD permanents to square the first comparison. This proves (4). Since R_t has positive diagonal, a strict predecessor bound gives strictness. □

4 Small-order closure and the all-orders consequence

We first evaluate the scalar losses exactly. If $3/2 \le t^2 \le 2$, then

$$d_m(t) = 2/t^2 \qquad (m \ge 1). \tag{7}$$

The $s = 0$ term is at most $2/t^2$, the $s = 1$ term equals it, and the successive ratios for $s \ge 1$ are at most $(3/2)/t^2 \le 1$. The sequence $t^k/(k+1)$ has increasing successive ratios $t(k+1)/(k+2)$, so its maximum is at an endpoint. In particular,

$$b_m(t) = \max\{1, t^m/(m+1)\}.$$

For $t = \sqrt{2}, m = 5$ both losses are one. For $t = 4/3, m = 6, 7$, $(4/3)^m \le m+1$, so $b_m = 1$ and $d_m = 9/8$. For $t = 13/10, m = 8$, $t^8 = 815730721/10^8 < 9$, so $b_8 = 1$ and $d_8 = 200/169$.

Proof of Theorem 1.1. Zero diagonal has already been settled. Normalize a positive-diagonal input to a correlation matrix. Rank at most four makes every order-five input singular, so Proposition 2.1 gives $c_5 = 3/5$ strictly. Lemma 3.1 at $m = 5, t = \sqrt{2}$ gives $c_6 = 3/5$. Its next three applications give

$$c_7 = \frac{3}{5}\frac{9}{8} = \frac{27}{40}, \qquad c_8 = \frac{27}{40}\frac{9}{8} = \frac{243}{320}, \qquad c_9 = \frac{243}{320}\frac{200}{169} = \frac{1215}{1352}.$$

All are less than one, and strictness propagates at each step. □

Proof of Corollary 1.2. For $a_\sigma = \prod_i a_{i,\sigma(i)}$ and $b_\sigma = \prod_i b_{i,\sigma(i)}$, Cauchy–Schwarz gives

$$|\operatorname{per}(A \circ B)| \le \Big(\sum_\sigma |a_\sigma|^2\Big)^{1/2}\Big(\sum_\sigma |b_\sigma|^2\Big)^{1/2} = \sqrt{Q(A)Q(B)}.$$

Theorem 1.1 supplies the paired bounds for orders five through nine; [4, Theorem 1.2] supplies them for all $n \ge 10$. The Schur product is PSD, so its permanent is real and nonnegative. For $n = 4$ use Rodtes' theorem [3]. For $n < 4$, apply that theorem to $A \oplus I_{4-n}$ and $B \oplus I_{4-n}$ and use multiplicativity of permanents under direct sums. Rodtes' theorem is unrestricted in rank, so the padding introduces no hypothesis problem.

If both diagonals are positive, $P(A), P(B) > 0$: for instance the Gram symmetric tensor of nonzero vectors is nonzero, since under its identification with a product of nonzero linear forms the polynomial ring has no zero divisors. The strict local bounds and Ma's $\rho_4 < 1$ therefore imply the asserted strict coefficient-one result. A zero diagonal in either factor gives the stated vanishing. □

Remark 4.1. The quantitative improvement at order seven is not essential to the qualitative completion. Starting with the old $192/245$ factor in [2], choose $t = 8^{1/7}$ and then $t = 9^{1/8}$. The resulting order-nine factor is

$$\frac{768}{245\, 8^{2/7}\sqrt{3}} < 1, \qquad 768^{14} < 245^{14} 8^4 3^7.$$

Thus the main increment is the controlled-loss use of an existing margin. The rational choices above give simpler and stronger estimates.

Dependencies and reproducibility

The new transfer and finite scalar comparisons are proved in full here. The all-orders conclusion explicitly reuses Rodtes' order-four theorem, the singular-five theorem of [2], and Ma's Theorem 1.2 for $n \ge 10$; it does not use Ma's tight-nine certificate or Li–Zhao–Luo's unrestricted through-six theorem. The reader package includes the unchanged singular-five certificate and its exact-integer verifier, the complete LaTeX source and a short exact check of the scalar comparisons. The numerical searches used during development are not proof inputs. The large-order theorem is cited rather than re-proved in this note; its independent reconstruction is a separate verification project.

AI assistance

AI tools assisted drafting and verification; the author is responsible for the content.

References

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