

Certified negative-band approximation for odd Weil forms

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Luminary Oddities

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Abstract

We construct certified finite approximations to the complete negative Fourier observation of an odd Weil form in its reciprocal-theta norm. Weighted Chebyshev features on a rigorous negative-symbol cover give a positive remainder with an explicit trace bound after exact orthogonal constraints. Discrete cosine orthogonality provides a degree-independent ridge charge. At arithmetic cutoff eleven, a 1366-feature construction admits ridge parameter 10^{-17} with complete trace error below $1.121 \cdot 10^{-14}$. A support-sensitive paired arithmetic-tail estimate yields a nonnegative addition to the localized positive reference while preserving the same negative observation. We derive the corresponding finite inverse conditions and a compact physical-to-weighted certificate transfer, retaining the complete covariance and tail costs.

1 Introduction

A signed Fourier form can be treated by retaining its complete positive part and approximating the negative observation. Quantitative use requires control in the specified weighted norm: all frequencies where the symbol can be negative, the entire physical line, exact constraints and the finite Gram metric must enter the error estimate together.

We use certified sign intervals and interpolate the sine observation locally with the true negative symbol as its weight. The error depends on component lengths rather than carrier frequencies. Weighted projection then gives a positive trace remainder. A ridge version controls its full coefficient cost through discrete cosine orthogonality, independently of the polynomial degrees. The actual M8 and M11 constructions demonstrate the reduction in sufficient finite input size.

For paired arithmetic forms a further compatibility step matters. On a compact physical support, the omitted paired diagonal has a controlled nonpositive sign. Its negative can be retained in the positive reference, with the original compact negative Fourier observation unchanged. A complete finite inverse condition then yields an original-norm lower bound with explicit approximation and arithmetic costs.

Projection, trace-ideal approximation and high-order quadrature are established tools; see Bornemann [1, Sections 2,5–6]. Positive-kernel approximation also has a substantial literature, including [2]. The contribution here is their complete signed-Weil realization, explicit raw-coordinate regularization cost, actual-symbol constructions and compatible support-sensitive paired-tail bounds. The original weighted normalization and exact-null framework are developed in [3, 4]; the present constructions keep that normalization throughout.

2 The original geometry and complete signed split

Use the positive even theta density

$$\psi(u) = \sum_{\ell \geq 1} (4\pi^2 \ell^4 e^{9u/2} - 6\pi \ell^2 e^{5u/2}) e^{-\pi \ell^2 e^{2u}}, \quad s(u)^2 = \frac{\psi(u)}{\cosh(u/2)}. \quad (1)$$

The series gives the function on $u \geq 0$; theta inversion gives its even extension. For real odd $h \in L^2(\mathbb{R})$ set $f = sh$ and impose

$$\int_{\mathbb{R}} s(u) h(u) \sinh(u/2) du = 0. \quad (2)$$

Let X be this closed weighted Hilbert space, optionally restricted by orthogonality to a specified finite span of genuine exact original null vectors. Write P_X for its actual orthogonal projection. The norm is $\|h\|_2 = \|f/s\|_2$.

Our Fourier convention is $\hat{f}(\xi) = \int_{\mathbb{R}} f(u) e^{-iu\xi} du$, with measure $d\xi/(2\pi)$. For integer $M \geq 2$ put

$$m_M(\xi) = \Re \frac{\Gamma'}{\Gamma}(1/4 + i\xi/2) - \log \pi - 2 \sum_{2 \leq n \leq M} \frac{\Lambda(n)}{\sqrt{n}} \cos(\xi \log n), \quad (3)$$

where Λ is the von Mangoldt function. Every retained prime power is included. Define $p_M = (m_M)_+$, $n_M = (-m_M)_+$ and

$$P_M[h] = \int_{\mathbb{R}} p_M(\xi) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi}.$$

This is a closed nonnegative form on its original logarithmic Fourier domain in X . If n_M is supported in $[-L, L]$, its exact real negative observation is

$$(C_M h)(\xi) = \sqrt{n_M(\xi)} \int_{\mathbb{R}} s(u) h(u) \sin(\xi u) du, \quad B = L^2((0, L), d\xi/\pi). \quad (4)$$

Oddness gives the factor $d\xi/\pi$, and $N_M = C_M^* C_M$ is the complete negative part. Thus the arithmetic head is

$$H_M = P_M - N_M. \quad (5)$$

The full constrained odd Weil form q_H retains every arithmetic shift. Its negative polar term vanishes by (2). On the original closed form domain it satisfies $q_H = H_M + W_{>M}$, with the complete bounded shift tail estimated in Section 7. Positive reference energy at every frequency remains in P_M .

Lemma 2.1 (Complete theta mass). *For $R \geq 0$ and $x_R = \pi e^{2R}$,*

$$S_0 := \int_{\mathbb{R}} s^2 \leq 16(\pi + 1)e^{-\pi} < 3, \quad S_{>R} := \int_{|u| > R} s^2 \leq 16(x_R + 1)e^{-x_R}. \quad (6)$$

Proof. For $u \geq 0$ set $r = \pi e^{2u}$. The positive atom bound and its geometric tail give $\psi(u) \leq (400/99)r^2 e^{u/2-r}$: consecutive majorants have ratio at most $16e^{-3\pi} < 1/100$. Dividing by $\cosh(u/2) \geq e^{u/2}/2$ yields $s^2 \leq 16\pi^2 e^{4u-\pi e^{2u}}$. Evenness and substitution of r give both bounds by integrating re^{-r} . $\square$

Main estimates and their application. The constructions in Sections 3–4 give finite approximants N_ρ to N_M with

$$0 \leq N_M - N_\rho \leq \tau_\rho I_X, \quad \tau_\rho = \tau_0(R) + \rho S_0 \sum_j \eta(r_j), \quad \eta(1) = 1, \quad \eta(r) = 2 \ (r \geq 2).$$

Here $\tau_0(R)$ is the explicit interpolation and physical-tail bound in (9); j indexes the negative-symbol cover intervals and r_j their polynomial feature counts. Thus the regularization charge is bounded by twice the number of intervals, times ρS_0 , independently of their polynomial degrees. The orthogonal approximation has error bound $\tau_0(R)$. If the finite inverse criterion (14) holds and the complete arithmetic tail has norm at most δ , Theorem 6.2 gives

$$q_H[h] \geq -(\kappa + \tau_\rho + \delta) \|h\|_2^2.$$

The proof first interpolates locally in frequency, integrates the error over the complete physical line, and uses discrete cosine Parseval to bound the joint interpolation coefficient norm. The inverse criterion then converts this approximation into a lower bound in the same weighted space. Section 5 supplies certified finite constructions; Sections 7–8 give the paired-tail and supported physical applications.

3 Adaptive one-sided trace approximation

Let intervals $I_j = [a_j, b_j]$ with disjoint interiors cover every positive frequency where $n_M > 0$. Set $\ell_j = b_j - a_j > 0$ and $\beta_j = \int_{I_j} n_M(\xi) d\xi / \pi$. For integers $r_j \geq 1$ and the affine map $t_j : I_j \rightarrow [-1, 1]$, use the raw feature columns

$$z_{j,k} = \mathbf{1}_{I_j} \sqrt{n_M} T_k(t_j), \quad 0 \leq k < r_j. \quad (7)$$

Let Z be their map into B , $D = Z^*Z$ and Π the orthogonal projection onto its finite range. Dependence on zero columns are handled by that true range: $\Pi = ZD^\dagger Z^*$.

Theorem 3.1 (Complete adaptive trace estimate). *With $N_{\text{fin}} = C_M^* \Pi C_M$, for every $R > 0$,*

$$0 \leq N_M - N_{\text{fin}}, \quad (8)$$

$$\text{Tr}(N_M - N_{\text{fin}}) \leq \tau_0(R) := \sum_j \beta_j \left[\frac{4S_0(\ell_j R/4)^{2r_j}}{(r_j!)^2} + S_{>R} \right]. \quad (9)$$

Consequently $N_M - N_{\text{fin}} \leq \tau_0(R) I_X$ and $\text{rank } N_{\text{fin}} \leq \sum_j r_j$.

Proof. For real u , interpolate $\sin(\xi u)$ at the r_j Chebyshev roots on I_j . The r_j th derivative is bounded by $|u|^{r_j}$, and the monic nodal product has supremum $2(\ell_j/4)^{r_j}$. Thus the uniform error is at most $2(\ell_j|u|/4)^{r_j}/r_j!$. Weighted orthogonal projection is no worse than this competitor. Integrating its square on $|u| \leq R$ gives the first term of (9). On $|u| > R$ use the zero competitor and $|\sin| \leq 1$. This bounds $\|(I - \Pi)C_M\|_{\text{HS}}^2$ on the ambient real odd space. Restriction to X is a right orthogonal contraction in that norm. Finally $N_M - N_{\text{fin}} = C_M^*(I - \Pi)C_M$ is positive; its trace is this squared Hilbert–Schmidt norm and bounds its operator norm. $\square$

The exact physical columns are $E_0 = C_M^* Z$, namely

$$(E_0)_{j,k}(u) = P_X \left[s(u) \int_{I_j} n_M(\xi) T_k(t_j(\xi)) \sin(\xi u) \frac{d\xi}{\pi} \right]. \quad (10)$$

They satisfy $N_{\text{fin}} = E_0 D^\dagger E_0^*$ and $\ker D \subseteq \ker E_0$, since $Zc = 0$ implies $C_M^* Zc = 0$.

4 Degree-independent ridge control

For $\rho > 0$ define

$$\Pi_\rho = Z(D + \rho I)^{-1}Z^*, \quad N_\rho = E_0(D + \rho I)^{-1}E_0^*.$$

These are the actual ridge operators in the raw coordinates (7). The positive contraction Π_ρ obeys the variational identity

$$\langle v, (I - \Pi_\rho)v \rangle = \min_a \{ \|v - Za\|^2 + \rho \|a\|^2 \}. \quad (11)$$

It follows by completing the square with $D + \rho I$.

Lemma 4.1 (Joint interpolation coefficient bound). *At the r roots $\theta_l = (l + 1/2)\pi/r$, interpolate values v_l with $|v_l| \leq 1$ by $\sum_{k < r} a_k T_k$. Then*

$$|a_0|^2 + \frac{1}{2} \sum_{k=1}^{r-1} |a_k|^2 = \frac{1}{r} \sum_{l=0}^{r-1} |v_l|^2 \leq 1.$$

Thus $\sum |a_k|^2 \leq \eta(r)$, where $\eta(1) = 1$ and $\eta(r) = 2$ for $r \geq 2$.

Proof. The finite cosine sums give mutually orthogonal columns $\cos(k\theta_l)$, with squared norms r for $k = 0$ and $r/2$ otherwise. For example use $2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$ and sum the geometric exponentials at these roots. Hence $a_0 = r^{-1} \sum v_l$ and $a_k = 2r^{-1} \sum v_l \cos(k\theta_l)$ for $k > 0$. Parseval in this orthogonal finite basis gives the identity. Twice its left side minus $|a_0|^2$ is the unweighted coefficient norm when $r \geq 2$; the degree-one case consists of a_0 alone. $\square$

Theorem 4.2 (Complete degree-independent ridge charge). *In the raw Chebyshev metric,*

$$0 \leq N_M - N_\rho, \quad \text{Tr}(N_M - N_\rho) \leq \tau_0(R) + \rho S_0 \sum_j \eta(r_j). \quad (12)$$

Also $\text{Tr } D \leq \sum_j r_j \beta_j$.

Proof. For fixed u with $|u| \leq R$, take in (11) the local interpolants of $\sin(\xi u)$. Lemma 4.1 bounds their joint coefficient norm by $\sum_j \eta(r_j)$, uniformly in carrier frequency, component width, degree and u . The interpolation error is as in Theorem 3.1. Outside $[-R, R]$ use the zero competitor. Integration against s^2 proves the ambient trace bound, and exact orthogonal compression preserves it. Positivity follows from $0 \leq \Pi_\rho \leq I$. Finally $|T_k| \leq 1$ on $[-1, 1]$ bounds each raw column norm by β_j . $\square$

The coefficient charge is at most $2\rho S_0 J$ for J components. This improves the separate-coordinate estimate $4\rho S_0 \sum_j r_j$. The result applies to the specified raw basis; column rescaling transforms both the coefficient penalty and the receiving metric by their actual congruence. In finite dimension it also gives the spectral bound $\text{cond}(D + \rho I) \leq 1 + \text{Tr}(D)/\rho$.

5 Actual M8 and M11 constructions

Our portable program encloses (3) on every closed rational cell of $[0, L]$. For $a = 1/4$, the convergent series

$$\Re \frac{\Gamma'}{\Gamma}(a + iy) = -\gamma + \sum_{k \geq 0} \left[\frac{1}{k+1} - \frac{k+a}{(k+a)^2 + y^2} \right]$$

shows monotonicity in $y \geq 0$. Its endpoint values enclose the digamma term on each cell. The cosine terms are enclosed on the whole cell by ball arithmetic. Every negative or uncertain cell is retained; adjacent retained cells are merged. The entire exterior is covered by

$$m_M(\xi) \geq \Re \frac{\Gamma'}{\Gamma}(1/4 + iL/2) - \log \pi - 2 \sum_{2 \leq n \leq M} \frac{\Lambda(n)}{\sqrt{n}} \quad (\xi \geq L). \quad (13)$$

The right side is freshly verified positive for $L = 4800$ at $M = 8$ and $L = 76800$ at $M = 11$. Every component mass is enclosed by summing whole-cell negative-part uppers. The degree allocator accepts only a directed interval comparison with the full trace formula.

M	Trace budget	Feature upper	Complete trace upper
8	10^{-12}	430	$2.261 \cdot 10^{-13}$
8	10^{-20}	577	$1.820 \cdot 10^{-21}$
8	10^{-30}	758	$2.035 \cdot 10^{-31}$
11	10^{-12}	1259	$1.591 \cdot 10^{-13}$
11	10^{-14}	1366	$1.636 \cdot 10^{-15}$
11	10^{-20}	1706	$1.687 \cdot 10^{-21}$
11	10^{-30}	2216	$1.231 \cdot 10^{-31}$

Table 1: Complete orthogonal-approximation bounds at $R = 2$. M8 uses width $1/32$; M11 uses width $1/16$. Feature counts bound rank from above.

The M8 cover has 52 components of total width $1535/32$ and final endpoint $39165/16$. Its negative mass is enclosed between 11.6389 and 14.0684 . The M11 cover has 167 components of total width $1781/16$ and final endpoint $158813/8$. The complete physical tail at $R = 2$ is below $8.883 \cdot 10^{-72}$. Every displayed cover is an enclosing cover, including uncertain cells.

For the 1366-feature M11 construction, all degrees exceed one, so $\sum \eta(r_j) = 334$. With $S_0 < 2.863591143$, the directed estimates are

$$\begin{aligned} \tau_0 &< 1.635911877 \cdot 10^{-15}, & \text{Tr } D &< 332.627, \\ \rho = 10^{-20} : & & \text{Tr}(N_{11} - N_\rho) &< 1.645476271 \cdot 10^{-15}, \\ \rho = 10^{-17} : & & \text{Tr}(N_{11} - N_\rho) &< 1.120030630 \cdot 10^{-14}. \end{aligned}$$

Thus the latter ridge fits the same complete $2 \cdot 10^{-14}$ budget. Its conservative condition upper is $3.327 \cdot 10^{19}$. The ridge charge is independent of degree; the approximation term still depends on each chosen degree through (9).

For a same-cover global Taylor comparison, the sufficient odd feature counts for the three M8 rows are 6659, 6663, 6669. For the M11 rows they are 53967, 53968, 53972, 53977. These compare sufficient constructions using the same certified support endpoint and $R = 2$; the adaptive method instead uses component lengths. The full rational endpoints, degrees, mass enclosures and local error terms are supplied in the data files.

6 Finite inverse criteria for the complete form

Lemma 6.1 (Positive-reference inverse criterion). *Let $P \geq 0$ be a closed form, $L = P + \kappa I$ with $\kappa > 0$, E a finite-column map and $B > 0$ a finite matrix. Then*

$$L \geq EB^{-1}E^* \iff E^*L^{-1}E \leq B.$$

For $B \geq 0$, restrict to its true range when $\ker B \subseteq \ker E$.

Proof. Both inequalities say that $L^{-1/2}EB^{-1/2}$ is a contraction. The form statement follows first on the form domain and then by closure. The restricted finite matrix is positive definite on its range. $\square$

Theorem 6.2 (Lower bound in the original weighted norm). *Suppose $q_H = H_M + W_{>M}$ with $\|W_{>M}\| \leq \delta$ on the same original recipient. Choose $N_\star = E_0 B_\star^{-1} E_0^*$ with $B_\star = D + \rho I$, or D on its range, and let $0 \leq N_M - N_\star \leq \tau_\star I$. If*

$$E_0^*(P_M + \kappa I)^{-1} E_0 \leq B_\star, \quad (14)$$

then $q_H[h] \geq -(\kappa + \tau_\star + \delta)\|h\|_2^2$.

Proof. Lemma 6.1 gives $P_M + \kappa I \geq N_\star$. Subtract the positive approximation remainder and the complete arithmetic tail in the same norm. The bounded differences preserve the original form domain. $\square$

For example, the complete M11 arithmetic bound in (16) below is less than $8 \cdot 10^{-13}$. At target allowance 10^{-10} and trace budget $2 \cdot 10^{-14}$, the matched shift is

$$\kappa = 10^{-10} - 8 \cdot 10^{-13} - 2 \cdot 10^{-14} = 9.918 \cdot 10^{-11}.$$

The sufficient finite condition is (14) with this κ and the actual raw metric $D + 10^{-17}I$.

7 Paired tails and a localized positive reference

Put $\Phi = -\psi'$, $\phi = \Phi > 0$ on $(0, \infty)$ and $f = \Phi g$, where g is even. The full odd weighted norm is $\mathcal{N}[g] = 2 \int_0^\infty \phi^2 s^{-2} |g|^2 = \|h\|_2^2$. Set $c_n = \Lambda(n)/\sqrt{n}$ and $b_n = \log n$. Expanding the paired edge squares on the smooth original core gives

$$\begin{aligned} E_n[g] &= 2c_n \int_0^\infty \phi(u)\phi(u+b_n)|g(u) - g(u+b_n)|^2 du \\ &\quad - c_n \int_0^{b_n} \phi(u)\phi(b_n-u)|g(u) - g(b_n-u)|^2 du \\ &= W_n[f] + \int_0^\infty D_n(u) d\mathcal{N}(u), \\ D_n(u) &= c_n s(u)^2 \frac{\phi(u+b_n) + \operatorname{sgn}(u-b_n)\phi(|u-b_n|)}{\phi(u)}. \end{aligned} \quad (15)$$

Here $W_n[f] = -c_n \int_{\mathbb{R}} f(u)[f(u+b_n) + f(u-b_n)] du$ is the original full odd arithmetic term. The bounded jointly extended expression on the right specifies the paired tail on the original domain.

Lemma 7.1 (Complete profile and shift bounds). *For $u > 0$, $us^2/\phi \leq 152$. For $M \geq 2$,*

$$\|W_{>M}\| \leq \sigma_M = \frac{16000\pi^2}{891}(M+1)^{3/2} \log(M+1) e^{-\pi(M+1)}. \quad (16)$$

When $r = \pi e^{2v} \geq 24$,

$$0 < \psi''(v) \leq 17\pi^4 e^{17v/2 - \pi e^{2v}}, \quad \phi'(v) < 0. \quad (17)$$

Proof. The profile details and complete near-origin scalar enclosure are given in Appendix A. For the far envelope, differentiation of (1) gives $\psi''(v) = e^{v/2} \sum_{\ell \geq 1} P_2(\ell^2 r) e^{-\ell^2 r}$, with $P_2(x) = 16x^4 - 112x^3 + 165x^2 - 75x/2$. For $x \geq 24$, $0 < P_2(x) < 16x^4$ and $\sum_{\ell \geq 2} \ell^8 e^{-r(\ell^2-1)} \leq 512e^{-72} < 1/16$. The first omitted term is $256e^{-3r}$ and later ratios are bounded by $(3/2)^8 e^{-5r} < 1/2$. This proves (17).

The stronger atom bound $s^2 \leq (800/99)\pi^2 e^{4|u| - \pi e^{2|u|}}$ follows as in Lemma 2.1. For $b = \log n$ put $x = e^{2|u|}$, $y = e^{2|u+b|}$, $z = \sqrt{xy}$. Then $z \geq n$, $x+y \geq 2z$, and $\sup_u s(u)s(u+b) \leq (800/99)\pi^2 n^2 e^{-\pi n}$. The two shifts cost at most twice this times c_n . For $n \geq 3$, the consecutive all-integer majorant $n^{3/2} \log n e^{-\pi n}$ has ratio below $(4/3)^{5/2} e^{-\pi} < 1/10$. Its geometric sum proves (16). $\square$

Theorem 7.2 (Support-sensitive diagonal tail). *For $R > 0$, $M \geq 2$, $N = M + 1$ and $A_R = \pi e^{-2R}$, assume $A_R N^2 \geq 24$. The even multiplier $d_{>M} = \sum_{n>M} D_n$ on $[-R, R]$ then satisfies*

$$-T_{M,R} \leq d_{>M} \leq 0, \quad (18)$$

$$T_{M,R} = 5168\pi^4 e^{-17R/2} (1 + N/38)(\log N) N^8 e^{-A_R N^2}. \quad (19)$$

The jointly extended tail has norm at most $T_{M,R} + \sigma_M$ there. If $M \geq e^{2R}$, its compressed shift part is zero.

Proof. For $n \geq N$, $\log n > R$, and the numerator of D_n is $\phi(\log n + u) - \phi(\log n - u)$. It is nonpositive by (17). The far envelope decreases where $r \geq 24$; the mean-value theorem and $us^2/\phi \leq 152$ give $|D_n| \leq 5168\pi^4 e^{-17R/2} (\log n) n^8 e^{-A_R n^2}$. For $t_n = (\log n) n^8 e^{-A_R n^2}$, $\log(t_{n+1}/t_n) \leq (8 + 1/\log 2)/n - A_R(2n+1) \leq -38/n$. The exact ratio is the product of three decreasing factors $\log(n+1)/\log n$, $(1+1/n)^8$ and $e^{-A_R(2n+1)}$. Thus all ratios for $n \geq N$ are at most $e^{-38/N}$, and $\sum t_n \leq (1 + N/38)t_N$. This proves (19), summing over every integer and hence every prime power. Uniform convergence of the diagonal and norm convergence of the shifts extend the paired identity from the original core. If $\log n > 2R$, support overlap is zero. $\square$

On the supported odd space with its exact polar row, let $V_{>M} = -d_{>M} \geq 0$. The paired head is

$$F_M = H_M - d_{>M} = (P_M + V_{>M}) - N_M, \quad q_H = F_M + W_{>M} + d_{>M}.$$

Its negative observation is still C_M . Additional constraints are imposed by their actual support-preserving projection.

Corollary 7.3 (Complete localized inverse criterion). *If $E_0^*(P_M + V_{>M} + \kappa I)^{-1} E_0 \leq B_*$, then on the original supported domain*

$$q_H[h] \geq -[\kappa + \tau_\star + T_{M,R} + \sigma_M] \|h\|_2^2.$$

For this supported recipient the trace estimate may use

$$\tau_\star \leq 4S_R \sum_j \beta_j \frac{(\ell_j R/4)^{2r_j}}{(r_j!)^2} + \rho S_R \sum_j \eta(r_j), \quad S_R = \int_{-R}^R s^2,$$

with the final term omitted for orthogonal projection.

Proof. Apply Lemma 6.1 to the new positive reference, then subtract the positive approximation remainder and the complete paired tail. The interpolation proof integrates only on $[-R, R]$ on this recipient. The added diagonal is bounded and preserves the original form domain. $\square$

Corollary 7.4 (Cofinal arithmetic schedule). *For $j \geq 1$ take $R_j = j \log 2/4$, $M_j = \lceil 16j2^{j/4} \rceil$. Then $T_{M_j, R_j} + \sigma_{M_j} < 2^{-j}$. If the complete supported inverse criteria in Corollary 7.3 hold with $\kappa_j = 2^{-j}$ and $\tau_{\star, j} \leq 2^{-j}$, then $q_H \geq 0$ on the original compact core and its form closure.*

Proof. Write $N_j = M_j + 1$. Then $16je^{R_j} \leq N_j \leq 18je^{R_j}$, $R_j < j/4$, $\log N_j < 5j$ and $A_{R_j} N_j^2 > 768j^2$. Formula (19) gives $T_{M_j, R_j} < K j^{10} e^{R_j/2 - 768j^2}$ with $K = 5168 \cdot 256 \cdot 10 \cdot 18^8 < e^{64}$. Its logarithm is less than $64 + (81/8)j - 768j^2 < -2j$. Similarly (16) gives $\log \sigma_{M_j} < 20 + (23/8)j - 48j < -2j$. Each term is less than 2^{-j-1} . For a fixed compact-core vector, all sufficiently large R_j contain its support. The localized bound then gives $q_H[h] \geq -3 \cdot 2^{-j} \|h\|^2$; take the limit and extend by the original closed form. $\square$

8 A compact physical-to-weighted application

Let a_R be the physical restriction of the original odd Weil form to $[-R, R]$, and let A_R be its associated physical operator. For multiplication J_s by s on this interval, the normalization is

$$q_H[h] = a_R[J_s h]$$

on the supported constrained form domain. Suppose the physical operator has a certificate $A_R \geq gI + UFU^*$, $g > 0$, $F \geq 0$. Since s is continuous and positive, $s_{\min} = \min_{[-R, R]} s > 0$. On the weighted constrained recipient the certificate implies $q_H \geq B_X$, where B_X is the compression of

$$B_{\text{full}} = gJ_s^2 + J_s UFU^* J_s.$$

This maps the physical lower form into the actual weighted norm.

Theorem 8.1 (Inherited strict supported inverse bounds). *Assume $\kappa \geq \sigma_M$ and set $R_{\text{full}} = (\kappa - \sigma_M)I + B_{\text{full}}$, $R_X = P_X R_{\text{full}} P_X|_X$. For normalized finite columns $V = E_0 B_{\star}^{-1/2}$, put $G_{\text{full}} = V^* R_{\text{full}}^{-1} V$. Then*

$$V^*(P_M + \kappa I)^{-1} V \leq G_{\text{full}}(I + G_{\text{full}})^{-1} < I.$$

If $K \geq \|V\|^2$ and $b = \kappa - \sigma_M + g s_{\min}^2$, the upper is at most $K/(K + b)I$. The same implication holds after adding the nonnegative localized diagonal $V_{>M}$ to the reference.

Proof. From $H_M = q_H - W_{>M} = P_M - N_M$ obtain $P_M + \kappa I \geq N_M + R_X \geq VV^* + R_X$. Inverse order and the finite-rank inverse identity yield the bound $G_X(I + G_X)^{-1}$, where $G_X = V^* R_X^{-1} V$. The variational supremum for the compressed inverse is over X ; enlarging to the supported ambient odd space gives $G_X \leq G_{\text{full}}$. The map $G \mapsto I - (I + G)^{-1}$ is operator monotone. Finally $R_{\text{full}} \geq bI > 0$, so $G_{\text{full}} \leq (K/b)I$. Adding a nonnegative reference decreases its inverse in form order. $\square$

All mixed entries of UFU^* are retained in this formula. Theorem 8.1 applies to any complete local source certificate at its stated support; it supplies an analytic application of the finite inverse criterion in the corresponding weighted space.

9 Reproduction

The accompanying source archive contains this manuscript, four portable programs, fixed result tables, citation metadata and reader instructions. With Python and the pinned python-flint version installed, run

```
python code/certify_negative_part.py --cutoff 8 \
--denominator 32 --output m8.json
python code/certify_negative_part.py --cutoff 11 \
--denominator 16 --output m11.json
python code/certify_ridge.py --pilot m11.json --output ridge.json
python code/verify_profiles.py --output profiles.json
python code/verify_paired_tail.py --output tails.json
```

The sign and trace calculations use 192-bit ball arithmetic on whole closed cells. The exterior ray is regenerated from (13). The scalar programs verify the explicit constants and representative schedule cases; the proofs above supply their uniform parameter and original-domain conclusions. The result files list the actual component degrees and directed enclosures used in the displayed constructions.

A Theta profile constants and origin cancellation

Set $u_0 = 1/400$ and $r = \pi e^{2u}$. On $[0, u_0]$ the complete second derivative has the first eight atoms $e^{u/2} \sum_{\ell=1}^8 P_2(\ell^2 r) e^{-\ell^2 r}$. The remaining absolute tail is bounded by $80(81\pi e^{2u_0})^4 e^{-81\pi}$: its first normalized term starts beyond 250, and later ratios are below $(10/9)^8 e^{-19\pi} < 1/2$. Full-interval ball evaluation of the eight terms plus this tail gives $\psi'' < -1$. Since $\psi'(0) = 0$, $\phi(u) \geq u$ there. The atom envelope gives $\psi(0) < 2$, so $us^2/\phi \leq 2$.

For $u \geq u_0$ write $Q(r) = 8r^2 - 30r + 15$. At $r_0 = \pi e^{1/200}$ both $Q(r_0)$ and $Q'(r_0)$ are positive. Every first-derivative atom has the sign giving $\phi(u) \geq e^{u/2-r} r Q(r)$. Combining this with the positive theta bound yields

$$e^{5u/2} \frac{s^2}{\phi} \leq \frac{800}{99\pi} \frac{r^2}{Q(r)} < 1024.$$

The rational function decreases for $r \geq r_0$, and its endpoint is enclosed by the profile program. Hence $us^2/\phi \leq 1024 \sup_{u>0} u e^{-5u/2} = 2048/(5e) < 152$. This proves the full profile bound, including the removable origin. It also proves $\phi > 0$ on $(0, \infty)$.

The cosine-coefficient identity and all finite operator inversions in the main argument use the complete joint Gram. In particular the trace error for ridge is $\text{Tr } C_M^*(I - \Pi_\rho) C_M$ as in (11); its projection analogue uses the exact orthogonal Π . Both yield one-sided positive remainders in their stated metrics.

References

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