Admissible kernel lifts with positive endpoint forms and nonreal Fourier zeros

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Abstract

We construct admissible Fourier kernels whose physical endpoint moment matrices are positive definite at every finite order while their Fourier transforms have nonreal zeros. The construction uses prescribed Fourier multipliers of positive finite atomic measures. A sufficiently broad hyperbolic-cosh base makes the translated mixture uniformly strictly log-concave while retaining an order-one entire Fourier transform. A global centered-moment and cumulant estimate also preserves strict concavity of $G(u) = \log \phi(\sqrt{u})$ and of $\log(S'(u)^2 - S(u)S''(u))$, where $S(u) = \phi(\sqrt{u})$. The required scales are explicit and conservative. An exact four-atom multiplier realizes this separation under all these shape conditions. All zeros can be confined to any prescribed horizontal strip; an interior physical parameter still has negative polynomial squares in both parities. Classical Turán consequences are stated with their coefficient normalization and do not imply real-zero closure. The construction strengthens a generic nonimplication within the full admissible class.

1 Introduction

Positivity conditions for entire Fourier transforms occur at several different levels: the shape of the original kernel, inequalities for its Taylor coefficients, positive-definite associated kernels, and reality of every zero. These conditions should not be conflated. A positive kernel has a positive-definite Fourier transform on the real line, but that observation does not force its complex zeros to be real. Even positivity of the first associated kernel is insufficient in general [1].

We study this distinction inside the full admissible class. Here a kernel ϕ is *admissible* if it is smooth, positive and even, strictly decreasing on $(0, \infty)$, and for some $\varepsilon > 0$ every fixed derivative satisfies

$$\phi^{(j)}(s) = O\big(e^{-|s|^{2+\varepsilon}}\big) \quad (|s| \longrightarrow \infty).$$

Our examples satisfy the stronger double-exponential decay condition. For a real entire function F, write

$$\mathcal{L}_1[F](x) = F'(x)^2 - F(x)F''(x).$$

The endpoint form considered below has Fourier density $\mathcal{L}_1[F]/4$.

The central construction is a finite-atomic convolution with a broad hyperbolic-cosh kernel. Its Fourier transform factors as

$$F_L(z) = 2LK_{iLz}(1) \sum_j w_j e^{ia_j z}.$$

The Bessel factor has only real zeros. Thus the Fourier multiplier of a positive atomic measure can introduce specified nonreal zeros while the broad base enforces strong kernel shape. The main

difficulty is global: the ratios between shifted cosh kernels are not uniformly close to one on the whole line. A compact perturbation argument would not control the tails.

We control the complete tilted mixture instead. An exact pair-variance identity bounds its ordinary logarithmic curvature. Centered-moment estimates then control the higher joint cumulants uniformly, with cost proportional to a single tilted mean rather than a power of that mean. An even-to-square-root differentiation identity and a positive hypergeometric coefficient comparison transfer these bounds to two shape levels. These are the reusable parts of the method. The scales are explicit sufficient bounds; they are intentionally conservative.

Principal separation theorem. The endpoint positivity and the interior failure occur for the same fully admissible kernel. The following result fixes the zero strip before the lifting parameters are chosen.

Theorem 1. *For every $h > 0$ there is an admissible kernel of the constructed type satisfying the ordinary logarithmic-curvature conclusion of Section 2 and both coordinate-shape conclusions of Section 3, whose Fourier transform has nonreal zeros but all its zeros lie in $|\operatorname{Im} z| < h$. Its physical endpoint matrices are positive definite at every finite order, while at some $0 < y < h$ the physical form has negative polynomial squares in both parities.*

Its proof is given in Section 4. The shape statements referred to here are the two explicit logarithmic shape inequalities proved in the intervening section. In particular, endpoint positive definiteness holds at every finite order, whereas the negative squares occur at an interior parameter in each parity. The scale of the atomic multiplier is chosen before the broad lifting scale.

Closest predecessors and the increment. The ordinary Laguerre inequality, its zero criteria, and the Bessel real-zero ingredients are classical [3, 2]. Csordas's close Example 3.12 [1] already supplies a positive, even, strictly log-concave Gaussian-polynomial kernel with a positive-definite first associated kernel and nonreal Fourier zeros. That example expressly omits the stronger admissibility decay condition. Our contribution is a systematic lift into the full admissible class, retaining an order-one transform and imposing two square-root-coordinate concavity levels by an explicit global cumulant estimate. The general insufficiency of the first Laguerre inequality is classical. Broad smoothing into log-concavity is also classical; the combination of admissibility and higher-shape estimates is the distinction here.

The author's previous theta paper [8] proves finite-order positivity and quantitative obstructions for a different, specific arithmetic kernel. Those results are credited and are not repackaged here as new theorems. The present paper addresses generic admissible kernels. It supplies a proof-route counterexample, not an off-line zero of the Riemann xi function and not a refutation of a theta-specific conjecture.

Organization and verification. We first prove the finite-atomic curvature and cumulant bounds, then the stronger coordinate-shape theorem. The Fourier section proves real-zero preservation of the base, the exact endpoint/interior identity, and the complete counterexample, including the passage from a negative density interval to polynomial squares. The final section separates classical coefficient consequences from the new construction and states its limits. The accompanying exact-arithmetic program reconstructs finite constants and algebraic controls; the analytic all-parameter proofs do not rest on numerical sampling.

2 A finite-atomic lift with global curvature

Let

$$\mu = \sum_{j=1}^{n} w_j \delta_{a_j} \tag{1}$$

be a nonzero finite positive even measure. Coincident locations are combined, so the a_j are distinct and $w_j > 0$. No probability normalization is required. Write

$$W = \sum_j w_j, \quad q = \frac{\max_j w_j}{\min_j w_j}, \quad D = \max_j a_j - \min_j a_j, \quad K = \binom{n}{2}, \quad T = Kq.$$

For $L > 0$ define

$$\phi_L(s) = \sum_j w_j \exp\left[-\cosh\left(\frac{s - a_j}{L}\right)\right], \qquad \Delta = \frac{D}{L}, \tag{2}$$

and put

$$C_L = \frac{T}{e}(e^{\Delta} - 1) + 2\sinh^2(\Delta/2). \tag{3}$$

Here admissibility means smoothness, strict positivity, evenness, strict decrease on $(0, \infty)$, and super-Gaussian decay of every fixed derivative. Our construction gives the stronger bound $\phi_L^{(k)}(s) = O(e^{-|s|^{2+\varepsilon}})$ for every fixed $k \geq 0$ and $\varepsilon > 0$, after a threshold depending on those parameters.

Theorem 2 (Global finite-atomic curvature)**.** *If $C_L < 1$, then ϕ_L is an admissible kernel with an entire extension and*

$$\frac{d^2}{ds^2} \log \phi_L(s) \leq -\frac{1 - C_L}{L^2} \qquad (s \in \mathbb{R}). \tag{4}$$

For $n \geq 2$, the explicit sufficient scale

$$L \geq 2D(T + 1) \tag{5}$$

gives $C_L < 1/2$. If $n = 1$, evenness forces $a_1 = 0$, $D = T = C_L = 0$, and every $L > 0$ works.

Proof. Set $t = s/L$, $d_j = a_j/L$ and

$$A_j(t) = \cosh(t - d_j), \quad B_j(t) = \sinh(t - d_j), \quad r_j(t) = \frac{w_j e^{-A_j(t)}}{\sum_i w_i e^{-A_i(t)}}. \tag{6}$$

Expectations in this proof use the actual t-dependent probabilities r_j. For $\ell(t) = \log \phi_L(Lt)$, exact differentiation gives

$$\ell'' = -\mathbb{E}A + \operatorname{Var} B, \qquad \operatorname{Var} B = \sum_{i<j} r_i r_j (B_i - B_j)^2. \tag{7}$$

The hyperbolic identity

$$(B_i - B_j)^2 = (A_i - A_j)^2 + 4\sinh^2((d_i - d_j)/2)$$

separates the potentially large component difference from a bounded offset.

At each fixed t let $A_{\min} = \min_j A_j \geq 1$ and $u_{ij} = |A_i - A_j|$. The inequality $\cosh(v + h) \leq e^{|h|} \cosh v$ implies

$$A_j \leq e^{\Delta} A_{\min}, \qquad u_{ij} \leq (e^{\Delta} - 1) A_{\min}, \qquad r_i r_j \leq q e^{-u_{ij}}. \tag{8}$$

For the last inequality, if $A_i \geq A_j$, the denominator in $r_i r_j$ is at least $w_j^2 e^{-2A_j}$; the other order is identical. Since $u^2 e^{-u} \leq u/e$ for $u \geq 0$, all K pairs contribute at most

$$\sum_{i<j} r_i r_j (A_i - A_j)^2 \leq \frac{T}{e}(e^{\Delta} - 1) A_{\min}.$$

The offset contributes at most $4\sinh^2(\Delta/2)\sum_{i<j} r_i r_j \leq 2\sinh^2(\Delta/2)$. Absorb this constant using $A_{\min} \geq 1$. Thus $\operatorname{Var} B \leq C_L A_{\min} \leq C_L \mathbb{E}A$, proving (4). Exponential suppression in (8) is essential: a crude variance bound growing as $(\mathbb{E}A)^2$ would not control the unbounded tails.

Evenness gives $(\log \phi_L)'(0) = 0$, and the strictly negative curvature implies $\phi_L'(s) < 0$ for $s > 0$. Positivity and the entire extension are immediate from the finite sum. Every derivative is a finite sum of a polynomial in $e^{s/L}, e^{-s/L}$ times a translated cosh exponential. Its doubly exponential tails imply the stated admissibility estimates.

For $n \geq 2$, $T \geq 1$, and (5) gives $\Delta \leq 1/[2(T+1)] \leq 1/4$. On this interval, $e^{\Delta} - 1 \leq 2\Delta$, $2/e < 1$, and $2\sinh^2(\Delta/2) \leq \Delta^2$; the latter follows from $\sinh u \leq u/(1-u^2)$. Consequently

$$C_L < T\Delta + \Delta^2 \leq \frac{T}{2(T+1)} + \frac{1}{4(T+1)^2} = \frac{1}{2} - \frac{2T+1}{4(T+1)^2} < \frac{1}{2}.$$

The singleton case follows directly from its centered base and requires no division by a support gap. □

The estimate is sufficient, not a sharp criterion for log-concavity of a mixture. Its usefulness here is uniformity in the real coordinate and its compatibility with retaining an exact finite Fourier multiplier.

3 Two logarithmic shape inequalities

Keep the atomic data and component notation of Section 2. Set

$$S_L(u) = \phi_L(\sqrt{u}), \qquad G_L(u) = \log S_L(u), \qquad \mathcal{T}_L(u) = S_L'(u)^2 - S_L(u) S_L''(u).$$

Primes on these functions are derivatives in u. Evenness gives their removable real-analytic extensions at $u = 0$. We will prove positivity of $\mathcal{T}_L$ and logarithmic concavity of this positive expression, in addition to logarithmic concavity of S_L itself.

Define the explicit integers

$$M_r = 2^r\{2(r-1)^{r-1} + 2^r\} \quad (2 \leq r \leq 8), \qquad \mathcal{B}_8 = 8^{16}\, 7!\, M_8. \tag{9}$$

Theorem 3 (Global root-coordinate and second-level shape)**.** *Suppose*

$$\Delta \leq 1/125, \qquad C_L < 1/10, \qquad \mathcal{B}_8 T\Delta \leq 1/4.$$

Then, for every $u \geq 0$,

$$G_L''(u) \leq -\frac{249}{4000L^4} < -\frac{1}{25L^4}, \qquad \mathcal{T}_L(u) > 0, \qquad (\log \mathcal{T}_L)''(u) < -\frac{1}{10L^4}. \tag{10}$$

The ordinary-coordinate bound is also $(\log \phi_L)'' < -9/(10L^2)$. For $n \geq 2$, one explicit sufficient scale for all these conclusions is

$$L \geq \max\{1,\ 125D,\ 10D(T+1),\ 4\mathcal{B}_8 TD\}. \tag{11}$$

For a single even atom, the direct conclusions hold at every $L > 0$.

The large scale in (11) is a conservative proof budget. The condition is sufficient. In particular, a smaller scale verified for the first root-coordinate inequality must not be assumed to satisfy the second-level conclusion.

3.1 Global centered moments and logarithmic derivatives

Lemma 4. *If $0 \le \Delta \le 1/4$, then for $k = 4, 8$ and every real t,*

$$|\ell^{(k)}(t) + \mathbb{E}A(t)| \le \mathcal{B}_8 T\Delta\,\mathbb{E}A(t), \qquad \ell(t) = \log \phi_L(Lt). \tag{12}$$

Proof. First, for any component-index random variable X and integer $r \ge 2$, conditional Jensen gives

$$\mathbb{E}|X - \mathbb{E}X|^r \le \sum_{i,j} r_i r_j |X_i - X_j|^r.$$

Let $u_{ij} = |A_i - A_j|$ and $v = 2\sinh(\Delta/2)$. The pair identity implies $|B_i - B_j| \le u_{ij} + v$. Use (8) and

$$u^r e^{-u} \le \left(\frac{r-1}{e}\right)^{r-1} u \le (r-1)^{r-1} u.$$

The ordered-pair factor two and $(u+v)^r \le 2^{r-1}(u^r + v^r)$ yield

$$\begin{aligned}\mathbb{E}|B - \mathbb{E}B|^r &\le 2^r T\{(r-1)^{r-1}(e^\Delta - 1)A_{\min} + v^r\} \\ &\le M_r T\Delta\,\mathbb{E}A \qquad (2 \le r \le 8).\end{aligned}$$

Here $e^\Delta - 1 \le 2\Delta$, $v \le 2\Delta$, $\Delta^r \le \Delta$, and $\mathbb{E}A \ge A_{\min} \ge 1$. The A-centered moment is at most $4T(r-1)^{r-1}\Delta\,\mathbb{E}A$, smaller than the same M_r bound. These estimates are global in t.

For completeness we pay the mixed cumulants without introducing a false power of $\mathbb{E}A$. Let $X_1, \dots, X_r$ each be a signed A or B, $Y_i = X_i - \mathbb{E}X_i$, and $H_r = M_r T\Delta\,\mathbb{E}A$. For any block C of a set partition of $\{1, \dots, r\}$, Hölder's inequality gives

$$\left|\mathbb{E}\prod_{i\in C} Y_i\right| \le \prod_{i\in C} (\mathbb{E}|Y_i|^r)^{1/r} \le H_r^{|C|/r}.$$

The product over all blocks is therefore at most H_r, even when it is smaller than one. Cumulants of order at least two are invariant under centering. Their partition formula consequently gives

$$|\operatorname{cum}_r(X_1, \dots, X_r)| \le r^r (r-1)!\, M_r T\Delta\,\mathbb{E}A. \tag{13}$$

There are at most r^r partitions and each coefficient $(|\pi| - 1)!$ is at most $(r-1)!$; counting vanishing singleton blocks only enlarges this bound.

Put $h_j(t) = -A_j(t)$. At a fixed real t, the exact finite-jet identity

$$\ell(t+z) - \ell(t) = \log \mathbb{E}_t \exp\left(\sum_{m\ge 1} h^{(m)}(t)\frac{z^m}{m!}\right)$$

shows that

$$\ell^{(k)}(t) = \sum_{\pi\in\Pi_k} \operatorname{cum}_{|\pi|}\big(h^{(|C_1|)}, \dots, h^{(|C_{|\pi|}|)}\big).$$

Here Π_k denotes labeled set partitions, and cum_1 is expectation. This formula includes all derivatives of the tilted probabilities. Local analyticity suffices; no uniformly zero-free complex strip for the mixture logarithm is assumed.

Each component derivative is $-A$ in even order and $-B$ in odd order. For even k, the one-block term is $-\mathbb{E}A$. Every remaining term is bounded by (13). For $r \le k \le 8$ use $r^r \le 8^8$, $(r-1)! \le 7!$, $M_r \le M_8$, and $|\Pi_k| \le 8^8$. This proves (12) with the full explicit budget (9). □

3.2 An exact positive-kernel change of coordinate

For any smooth even ℓ, and $t = \sqrt{u}/L > 0$, one has

$$\frac{d^j}{du^j}\ell(\sqrt{u}/L) = \frac{1}{L^{2j}2^{2j-1}(j-1)!}\int_0^1 (1-v^2)^{j-1}\ell^{(2j)}(vt)\,dv, \qquad j \geq 1. \tag{14}$$

For $j = 1$ this is the fundamental theorem, since $\ell'(0) = 0$. Differentiate in $u = L^2t^2$ and integrate by parts using $[(1-v^2)^j]' = -2jv(1-v^2)^{j-1}$. The boundary at $v = 1$ vanishes by the weight and that at $v = 0$ by evenness. The coefficient is multiplied by $1/(4jL^2)$, proving the induction. Thus an entire power series for ℓ is not required. At the real-analytic removable endpoint,

$$G_L^{(j)}(0) = \frac{j!}{(2j)!\,L^{2j}}\ell^{(2j)}(0).$$

Proof of Theorem 3. The ordinary curvature follows from Theorem 2. Put $d = \Delta/2 \leq 1/250$. Evenness centers the extreme atoms, so $|d_j| \leq d$. The elementary cosh ratio bound gives, globally,

$$e^{-d}\cosh t \leq \mathbb{E}A(t) \leq e^d \cosh t.$$

Lemma 4 and the quarter budget imply that $\ell^{(4)}, \ell^{(8)}$ are negative, with magnitudes between $(3/4)\mathbb{E}A$ and $(5/4)\mathbb{E}A$. The kernels in (14) are nonnegative. Comparing with $H(u) = -\cosh(\sqrt{u}/L)$ therefore yields

$$|G_L''| \geq (3/4)e^{-d}|H''|, \qquad |G_L''''| \leq (5/4)e^d|H''''|, \qquad G_L'', G_L'''' < 0. \tag{15}$$

These comparisons include $u = 0$ by continuity.

Write $z = u/(4L^2)$ and $F_b(z) = {}_0F_1(;b;z) = \sum_{m\geq 0} z^m/[m!(b)_m]$. Differentiating the cosh series gives

$$H'' = -\frac{F_{5/2}(z)}{12L^4}, \qquad H'''' = -\frac{F_{9/2}(z)}{1680L^8}.$$

For $z \geq 0$ coefficientwise comparison gives $F_{9/2}(z) \leq F_{5/2}(z)$ and $F_{5/2}(z) \geq 1$. Consequently

$$\frac{|H''''|}{2H''^2} \leq \frac{3}{70}, \qquad \frac{|G_L''''|}{2G_L''^2} \leq \frac{20}{9}e^{3d}\frac{3}{70} \leq \frac{500}{5187} < \frac{1}{10}.$$

The last rational bound uses $e^{3d} \leq e^{3/250} \leq 250/247$.

Also (15) gives $|G_L''| \geq (3/4)(249/250)/(12L^4) = 249/(4000L^4)$.

Since $S_L > 0$ and $G_L'' < 0$, the quantity $\mathcal{T}_L = -S_L^2 G_L''$ is positive. Exact differentiation now gives

$$(\log \mathcal{T}_L)'' = 2G_L'' + \frac{G_L''''}{G_L''} - \left(\frac{G_L'''}{G_L''}\right)^2 < -\frac{9}{5}|G_L''| \leq -\frac{2241}{20000L^4} < -\frac{1}{10L^4}.$$

All logarithms and derivatives here have their real-analytic values at $u = 0$, where G_L'' stays strictly negative. This proves every conclusion of (10) on the closed half-line.

For the sufficient scale, $L \geq 125D$ pays the diameter condition, and $L \geq 4\mathcal{B}_8 TD$ pays the derivative budget. For $n \geq 2$, $L \geq 10D(T+1)$ gives

$$C_L < T\Delta + \Delta^2 \leq \frac{T}{10(T+1)} + \frac{1}{100(T+1)^2} = \frac{1}{10} - \frac{10T+9}{100(T+1)^2} < \frac{1}{10}.$$

For a singleton all centered moments vanish, ℓ is a constant minus $\cosh t$, and the same base comparison proves the conclusions directly for every $L > 0$. □

The proof compares logarithmic derivatives, not the undifferentiated mixture with the base. Such a uniform relative function comparison would fail in the far tails. The global pair suppression, centered Hölder bound and finite cumulant budget are what permit the additional shape condition to coexist with the Fourier obstructions established next.

4 Fourier factorization, endpoint forms and interior failure

The analytic estimates above concern the kernel itself. We now retain its Fourier factor and distinguish endpoint moment positivity from positivity at all physical parameters.

4.1 The Bessel base and its zeros

Set

$$B_L(z) = \int_{\mathbb{R}} e^{-\cosh(s/L)} e^{izs}\, ds = 2LK_{iLz}(1), \qquad F_L(z) = \int_{\mathbb{R}} \phi_L(s) e^{izs}\, ds = B_L(z)p(z). \tag{16}$$

The first identity is the standard integral for the modified Bessel function; the second follows directly by Fourier translation. Complete double-exponential decay justifies complex differentiation at every finite z. Both functions are real even entire when the atomic measure is even.

For completeness their order is one, not two. If $|z| \le R$, the tails of each translated base are bounded by $\exp(-ce^{|s|/L} + R|s|)$ with a positive constant c depending only on the fixed translates. Substitution $v = ce^{|s|/L}$ bounds the integral by a constant times $c^{-LR}\Gamma(LR)$, with a harmless compact contribution. The elementary bound $\Gamma(r) \le 1 + \lceil r \rceil!$ for $r \ge 1$ proves $\log M(R) = O(R\log(R+2))$. On the positive imaginary axis there is no cancellation: all translates and weights are positive. Restricting the integral to a unit interval next to $s = -L\log R$ gives $\log F_L(iR) \ge LR\log R - O(R)$ as $R \to \infty$. Thus the order is exactly one. Constants in these growth estimates need not be uniform over the atom configurations.

Lemma 5. *Every zero of B_L is real, and $\mathcal{L}_1[B_L](x) \ge 0$ for every real x.*

Proof. The real-zero assertion is classical [2]; a short spectral proof fixes its domain. If $K_{iLz}(1) = 0$, put $v(t) = K_{iLz}(e^t)$ for $t \ge 0$. The Bessel equation gives

$$-v'' + e^{2t}v = L^2z^2v, \qquad v(0) = 0.$$

The integral representation, or its elementary large-argument estimates, gives superexponential decay of v and v' at infinity for every fixed complex order. Multiplying by $\overline{v}$, integrating and passing to infinity yields

$$L^2z^2 \int_0^\infty |v|^2\, dt = \int_0^\infty \left(|v'|^2 + e^{2t}|v|^2\right) dt > 0.$$

The function is not identically zero. Therefore z^2 is positive real and z is real. Also $B_L(0) > 0$.

Hadamard factorization for an even real entire function of order one now gives

$$B_L(z) = B_L(0)\prod_j (1 - z^2/\rho_j^2), \qquad \rho_j > 0, \quad \sum_j \rho_j^{-2} < \infty,$$

with multiplicities. The paired genus-one exponential factors cancel, and the remaining linear exponential is constant by evenness. For real x away from these zeros,

$$\frac{\mathcal{L}_1[B_L](x)}{B_L(x)^2} = -\frac{d^2}{dx^2}\log|B_L(x)| = \sum_j \left(\frac{1}{(x-\rho_j)^2} + \frac{1}{(x+\rho_j)^2}\right) \ge 0.$$

Locally uniform convergence justifies differentiation. Continuity covers the zeros themselves. □

4.2 The exact physical transform

For any of our lifted kernels write $\phi = \phi_L$, $F = F_L$ and define

$$H_y(s) = \frac{\sinh(2ys)}{2ys}, \quad H_0 = 1,$$
$$J_y(t) = \int_0^\infty s^2 H_y(s)\phi(s+t/2)\phi(s-t/2)\,ds, \qquad g_y(x) = \int_{\mathbb{R}} J_y(t)e^{ixt}\,dt. \tag{17}$$

The continuous interpretation is used when $ys = 0$. Put

$$m_{2j}(y) = (-1)^j J_y^{(2j)}(0), \quad m_{2j+1}(y) = 0, \qquad \mathsf{H}_N(y) = \left(\frac{m_{r+s}(y)}{J_y(0)}\right)_{r,s=0}^N.$$

All integrals and finite derivatives exist; $J_y(0) > 0$ for finite $y \geq 0$. The integer N is polynomial degree, not a parity-block dimension.

Lemma 6. *With the Fourier convention in* (17),

$$g_y(x) = \frac{\partial_y |F(x+iy)|^2}{8y} \quad (y > 0), \qquad g_0(x) = \frac{\mathcal{L}_1[F](x)}{4}. \tag{18}$$

For every real polynomial P,

$$\sum_{r,s} P_r P_s m_{r+s}(y) = \frac{1}{2\pi}\int_{\mathbb{R}} P(x)^2 g_y(x)\,dx. \tag{19}$$

Proof. Introduce the cosh-weighted kernel

$$C_y(t) = \int_0^\infty \cosh(2ys)\phi(s+t/2)\phi(s-t/2)\,ds.$$

Evenness makes this half of the whole-line center integral. The change of variables $a = s + t/2$, $b = s - t/2$ has absolute Jacobian one. Expanding the cosh gives $\widehat{C}_y(x) = |F(x+iy)|^2/2$. Since $\partial_y C_y = 4yJ_y$, the first formula follows. Taylor expansion of the real even entire F at a real point gives

$$|F(x+iy)|^2 = F(x)^2 + y^2\{F'(x)^2 - F(x)F''(x)\} + O(y^4),$$

which proves the continuous endpoint formula.

Here are explicit domination facts for these operations and inversion. For fixed finite y, shift the s contour in $F(x+iy)$ by $i\,\mathrm{sgn}(x)\pi L/4$. Each translated base retains a positive real cosh coefficient: $\mathrm{Re}\cosh((s-a_j)/L + i\pi/4) = \cosh((s-a_j)/L)/\sqrt{2}$. Vertical end segments vanish. The same estimate pays the linear and quadratic factors needed for F' and F''. Thus

$$|F(x+iy)| + |F'(x+iy)| + |F''(x+iy)| \leq C_y e^{-\pi L|x|/4}.$$

The constants can be made uniform on any compact set of y values. Consequently g_y and g_0 have exponential decay and every absolute moment. The complete product integrals and their fixed finite derivatives are dominated by the double-exponential tails, so Fubini, differentiation and Fourier inversion apply. Differentiating the inverse Fourier formula at zero gives (19). Odd moments vanish by evenness. □

Proposition 7. *If $\mathcal{L}_1[p](x) \geq 0$ for all real x, then $\mathsf{H}_N(0)$ is positive definite for every finite N.*

Proof. The elementary product identity

$$\mathcal{L}_1[B_L p] = B_L^2 \mathcal{L}_1[p] + p^2 \mathcal{L}_1[B_L]$$

and Lemma 5 give $g_0 \geq 0$. It is positive on an interval: $B_L'(0) = 0$, $B_L''(0) = -\int_{\mathbb{R}} s^2 e^{-\cosh(s/L)} ds < 0$, and $p(0) = \sum_j w_j > 0$. Hence $\mathcal{L}_1[F](0) > 0$, even if $\mathcal{L}_1[p]$ vanishes identically. Every nonzero polynomial has a strictly positive squared integral against this continuous nonnegative density. Formula (19) and the positive normalization $J_0(0)$ prove strict definiteness at all finite orders. □

4.3 Nonreal zeros and negative polynomial squares

Lemma 8. *Let g be real, even and continuous with $|g(x)| \leq Ce^{-c|x|}$, $c > 0$. If g is strictly negative on an interval away from zero, then each polynomial parity contains a real polynomial P with $\int P^2 g < 0$. Coefficients may be taken rational. In fact the negative index is unbounded in each parity.*

Proof. Polynomials are dense in $L^2(|g|\,dx)$. To see this without presuming positivity of the signed form, suppose h is orthogonal to every polynomial there. Cauchy–Schwarz shows that $h|g|\,dx$ is a finite signed measure with exponential moments in a strip of positive width. Its Fourier transform is analytic in that strip and every derivative at zero vanishes. The identity theorem and uniqueness of Fourier transforms give $h|g| = 0$ almost everywhere. This proves density.

Choose a nonzero real continuous function supported inside the negative interval and its reflection, with either desired parity. Its signed squared integral is negative. Approximate it in $L^2(|g|\,dx)$ by polynomials and project to that parity. The projection is a contraction because $|g|$ is even, and convergence implies convergence of signed quadratic integrals. A sufficiently close polynomial has strict negative integral. Continuity of its finite moment form permits rational coefficient approximation. For any prescribed finite dimension, start with disjointly supported parity functions in the negative region and approximate the entire finite basis; its negative Gram matrix persists under sufficiently small perturbation. The maximum polynomial degree of that finite set is finite. This proves unbounded negative index, not a numerical first-failure degree. □

If F has a nonreal zero $z_0 = c + ib$, $c > 0$, $b > 0$, factor $F(z) = (z - z_0)^m U(z)$ with $U(z_0) \neq 0$. At $x = c$, for $y < b$ near b,

$$\partial_y |F(c+iy)|^2 = -2m(b-y)^{2m-1}|U(c+iy)|^2 + (b-y)^{2m}\partial_y |U(c+iy)|^2 < 0.$$

Continuity yields a closed negative frequency interval for this fixed $y > 0$. The conclusion places the violation at a suitable height. For an even real F, reflection supplies the other interval. Lemmas 6 and 8 therefore give finite negative physical moment directions in both parities. This uses the complete lifted kernel, not a first-atom truncation or a numerical plot.

4.4 An exact counterexample and arbitrary zero strips

Proof of the principal separation theorem.

Proof. Begin with atoms at ± 2 of weight $1/2$ and at ± 4 of weight $1/10$. Then

$$p(z) = \cos(2z) + \frac{1}{5}\cos(4z), \qquad \mathcal{L}_1[p](x) = 4\left\{\frac{29}{25} + \frac{2}{5}\cos^3(2x) + \frac{3}{5}\cos(2x)\right\} \geq \frac{16}{25}.$$

The cubic in $v = \cos(2x)$ is increasing on $[-1, 1]$; its minimum proves the last inequality exactly. The equation $2v^2/5 + v - 1/5 = 0$ has one root in $(-1, 1)$ and the root $-(5 + \sqrt{33})/4 < -1$. Thus the nonreal zeros include

$$z_0 = \frac{\pi}{2} + i\beta, \qquad \beta = \frac{1}{2} \operatorname{arcosh} \frac{5 + \sqrt{33}}{4} \in (0, 1).$$

The upper bound follows from $1 < (5 + \sqrt{33})/4 < 3 < \cosh 2$. All nonreal zeros of this multiplier have imaginary height $\pm\beta$. Choose the explicit sufficient lifting scale in the shape theorem. For these atoms $D = 8$, $Kq = 30$, and, with the integer budget B_8 defined there, one convenient choice is

$$L = 3481 + 960B_8.$$

It exceeds 1, $125D$, $10D(Kq + 1)$ and $4B_8(Kq)D$. The Bessel factor introduces only real zeros and cannot cancel a nonreal zero of p. Proposition 7 gives all-order endpoint positivity; the local zero argument gives interior failure.

For an arbitrary strip, a direct general bound avoids a selected-zero quantifier error. For distinct atoms $a_1 < a_2 < \cdots < a_n$, $n \geq 2$, put $W = \sum w_j$, $d = a_2 - a_1$ and

$$Y = \frac{1 + \log(W/w_1)}{d}.$$

For $\operatorname{Im} z = y \geq Y$, all terms other than the a_1 term, divided by it, have total modulus at most $[(W - w_1)/w_1]e^{-dy} < 1$. Thus there are no zeros there, and evenness gives the same lower half-plane bound. Rescale the atom locations by any $\lambda > Y/h$. The new multiplier is $p(\lambda z)$: it retains the sign of $\mathcal{L}_1[p]$, and *all* zero heights are divided by λ. Only after this rescaling choose the sufficient L for the new atom span. Every nonreal multiplier zero persists in F_L, all Bessel zeros remain real, and the preceding endpoint and interior arguments apply. The strip parameter specifies a different member of the family. □

The scale is an explicit existence bound, not an efficient numerical choice. The second-level assertion uses the stated sufficient scale; smaller scales are not inferred to satisfy it from finite probes.

5 Moment inequalities and the scope of the construction

The shape statements give classical consequences for a normalized coefficient sequence. These are distinct from the physical endpoint moment forms. For any lifted kernel satisfying the first- and second-level shape hypotheses, put

$$b_k = \int_{\mathbb{R}} s^{2k} \phi(s)\, ds, \qquad \gamma_k = \frac{k!}{(2k)!} b_k, \qquad \Psi(u) = \sum_{k \geq 0} \gamma_k \frac{u^k}{k!}.$$

All these moments are finite and positive. With the Fourier convention used here, $F(z) = \Psi(-z^2)$. Using half-line moments instead multiplies the whole sequence by one common positive constant and changes none of the signs below. The factorial normalization must be retained.

Define the ordinary and double Turán differences by

$$T_k = \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \quad (k \geq 1), \qquad E_k = T_k^2 - T_{k-1}T_{k+1} \quad (k \geq 2).$$

The ordinary moment theorem recalled and proved in [4, Theorem 2.4(a)] gives $T_k > 0$ from strict concavity of $\log \phi(\sqrt{u})$. Under the additional strict concavity of $\log(S'^2 - SS'')$, with $S(u) = \phi(\sqrt{u})$, [4, Theorem 2.4(b)] gives

$$E_k \geq 0 \qquad (k \geq 2). \tag{20}$$

These are direct applications of classical theorems to the admitted lifted kernels, not new moment-inequality principles.

There is also a distinct cubic consequence. The higher Turán theorem of Dimitrov–Lucas [5, Theorem 1], together with ordinary Turán positivity, makes every shifted cubic Jensen polynomial

$$J_{3,n}(x) = \gamma_n + 3\gamma_{n+1}x + 3\gamma_{n+2}x^2 + \gamma_{n+3}x^3, \qquad n \geq 0,$$

hyperbolic, meaning that all its zeros are real. The cubic-discriminant inequality in that theorem is a different expression from (20). Neither statement is an assertion of hyperbolicity at every Jensen degree, of infinite iterated log-concavity, or of membership in the Laguerre–Pólya class.

For the multiplier with nonreal zeros constructed above, the resulting F is not in that class, despite these ordinary, cubic and double Turán consequences and the positive endpoint forms at every finite degree. The kernel shape statements concern derivatives in $u = s^2$; the Turán statements concern the normalized Taylor coefficients of Ψ; the endpoint forms concern the Fourier density $\mathcal{L}_1[F]/4$. Identifying these three different objects would give an invalid implication.

5.1 Relation to the admissible-kernel literature

Csordas [1, Definition 1.2 and Example 3.12] gives the close example $e^{-s^2}(15+s^2+s^4)$. It is positive, even and strictly log-concave; its first associated kernel is positive definite while its Fourier transform has nonreal zeros. Thus endpoint positivity and its failure to imply real-zero closure are established phenomena. That example explicitly fails the admissibility decay requirement $\phi^{(j)}(s) = O(e^{-|s|^{2+\varepsilon}})$ for some $\varepsilon > 0$ and all fixed j. Its transform has order two. The present finite-atomic lift pays that stronger decay condition, has an order-one transform, and enforces the two square-root-coordinate shape levels at an explicit sufficient scale. It does not invalidate Csordas's all-associated-index criterion: some higher associated kernel must fail.

The scope of the strengthening is therefore a constructive nonimplication inside the full admissible class, with additional shape controls. It does not solve the characterization problem in [1, Open Problem 3.9], and it does not refute the theta-specific second-level conjecture posed in [4, Problem 3.3]. A recent preprint of Planat–Solé [7] reports a proof of that specific theta-kernel condition, using a local directed certificate together with theta-tail and elliptic continuations, and derives the associated double Turán consequence. Its object is the actual Riemann theta density; the present generic construction concerns a different sufficiency question.

The ordinary Bessel real-zero mechanism, Fourier convolution and Laguerre product identity remain classical ingredients [2, 3].

Broadening a finite location mixture until it becomes log-concave is also an established mechanism. For example, Huggins–Zou [6, Section 7] record the strong-log-concavity threshold for an equal-weight two-Gaussian mixture. That Gaussian example supplies useful context; it does not give the double-exponential admissibility, order-one transform or two-level shape estimates proved here.

5.2 Generic separation and related theta results

The Riemann theta density is a specific arithmetic and modular object. The author's earlier theta results [8] concern its finite-order physical positivity, stronger-cone obstructions and negative-band

polynomial transfer. The present construction supplies generic kernels and a self-contained endpoint-to-interior separation.

Assistance. Research and manuscript preparation used AI assistance. The analytic arguments and finite supplements are supplied for inspection.

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