A Weyl Law for Rapidly Twisted Polar-Strip Tubes

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October 22, 2026

Abstract

We determine the leading eigenvalue-counting asymptotic for a class of infinite-volume Dirichlet tubes confined by diverging twisting. The planar cross-section is a compact annular polar strip with positive C^1 angular width $\alpha(r) < 2\pi$, and the twisting speed is $c(1+s^2)^{p/2}$, where $c, p > 0$. We prove $N(\Lambda) \sim C\Lambda^{3/2+1/(2p)}$ and compute C explicitly in terms of $\zeta(1+1/p)$ and $\int r\alpha(r)^{1+1/p}\,\mathrm{d}r$. The proof retains the angular Dirichlet modes while treating the radial and longitudinal variables semiclassically. A uniform Dirichlet-box comparison obtained from quasiperiodic phase averaging and IMS localization controls the artificial longitudinal boundaries. A separate tail estimate and absolute-coefficient freezing justify the global limit. The result gives a concrete-class answer to the Weyl-asymptotics question for tubes with asymptotically diverging twisting.

Keywords: Dirichlet Laplacian; twisted waveguide; Weyl law; infinite-volume domain; eigenvalue asymptotics.
2020 Mathematics Subject Classification: 35P20, 35J25, 81Q10.

1 Introduction and main result

An unbounded Euclidean domain can have a Dirichlet Laplacian with compact resolvent even when its volume is infinite. Twisting provides such a confinement mechanism without shrinking the area of the cross-section. Krejčiřík proved discreteness for a class of tubes with asymptotically diverging twisting and explicitly asked for Weyl-type asymptotics [5]. Barseghyan and Khrabustovskyi subsequently proved eigenvalue-moment upper estimates [2]. Their Proposition 4.1 computes the large-energy asymptotic of the right-hand side of an upper bound; it does not identify a matching asymptotic for the actual counting function. The counting question is retained as Open Problem 8 in [6].

In this paper we obtain an exact leading counting law for annular polar-strip cross-sections and a smooth power-growth twisting speed. The coefficient involves the full sequence of angular Dirichlet levels, not only its lowest level. Consequently it depends on a nonlinear functional of the angular width rather than only on cross-sectional area. The elementary tools are variational bracketing, Fourier decomposition, phase averaging and IMS localization. Their uniform use is necessary because the twisting speed itself is of order $\sqrt{\Lambda}$ in a substantial part of the region contributing to $N(\Lambda)$.

Earlier eigenvalue asymptotics for twisted waveguides include the accumulation of bound states at a finite essential-spectrum threshold when the twisting speed approaches a constant [3]. That regime differs from the compact-resolvent, high-energy limit considered here. The retention of some variables as discrete spectral modes is consistent with the operator-valued viewpoint on nonclassical Weyl asymptotics discussed in [4, Section 3.2.1]. We give the localization and counting comparisons needed for the present geometry explicitly. Related comparisons with horn-shaped domains appear in the analysis of shrinking spiral regions [1, Section 4]. Thus discrete transverse-mode counting is established nonclassical-asymptotic

machinery; the result here concerns its uniform implementation and explicit actual counting law for the twisted geometry (1.4).

Fix $0 < a < b < \infty$. Let $l, u \in C^1([a,b];\mathbb{R})$, and suppose that, for some $\delta > 0$,

$$0 < \alpha(r) := u(r) - l(r) \le 2\pi - \delta \qquad (a \le r \le b). \tag{1.1}$$

Set

$$\omega = \{(r\cos t, r\sin t) : a < r < b,\ l(r) < t < u(r)\}. \tag{1.2}$$

The real-valued functions l, u specify continuous lifts of the angular endpoints; they need not lie in a single conventional branch interval. The bound in (1.1) ensures injectivity at each fixed radius. For $c, p > 0$ define

$$\beta(s) = c(1+s^2)^{p/2}, \qquad \theta(s) = \int_0^s \beta(v)\,\mathrm{d}v. \tag{1.3}$$

The associated tube is

$$\Omega = \{(r\cos(t+\theta(s)), r\sin(t+\theta(s)), s) : a < r < b,\ l(r) < t < u(r),\ s \in \mathbb{R}\}. \tag{1.4}$$

Let H_Ω be the Friedrichs Dirichlet Laplacian, namely the self-adjoint operator associated with the closure of $\int_\Omega |\nabla v|^2\,\mathrm{d}x$ on $C_c^\infty(\Omega)$. We count eigenvalues with multiplicity and use the convention

$$N_\Omega(\Lambda) = \operatorname{Tr} \mathbf{1}_{(-\infty,\Lambda)}(H_\Omega). \tag{1.5}$$

Theorem 1.1. *Under* (1.1)–(1.3)*, H_Ω has compact resolvent and*

$$N_\Omega(\Lambda) \sim C_\omega \Lambda^{d_p}, \qquad d_p = \frac{3}{2} + \frac{1}{2p}, \qquad \Lambda \longrightarrow \infty, \tag{1.6}$$

where

$$C_\omega = \frac{p\,\zeta(1+1/p)c^{-1/p}}{\pi^{2+1/p}(p+1)(3p+1)} \int_a^b r\alpha(r)^{1+1/p}\,\mathrm{d}r. \tag{1.7}$$

For the annular sector $l = 0$, $u = \alpha \in (0, 2\pi)$, the constant is

$$C_\alpha = \frac{p\,c^{-1/p}}{\pi(p+1)(3p+1)} \left(\int_a^b r\,\mathrm{d}r\right) \left(\frac{\alpha}{\pi}\right)^{1+1/p} \zeta(1+1/p). \tag{1.8}$$

In particular, $p = 1$ gives

$$N_\Omega(\Lambda) \sim \frac{\alpha^2(b^2-a^2)}{96\pi c}\Lambda^2. \tag{1.9}$$

The volume of every tube in the theorem is infinite, since the coordinate Jacobian is r and the cross-section has fixed positive area. An ordinary finite-volume three-dimensional Weyl formula is therefore inapplicable.

We first prove the sector case, including an auxiliary variant with Neumann radial faces. We then recover the varying angular boundaries by a fixed-mesh inner and outer sector approximation. Thus the theorem does not assume separation of a variable-boundary operator. The hypotheses exclude pinching angular widths, contact with the rotation axis and multiple angular components at a fixed radius.

2 The pulled-back form and confinement

For a sector write $\mathcal{Q} = (a,b)\times(0,\alpha)\times\mathbb{R}$ and use coordinates (r,t,s). The map in (1.4) is injective: equality of the third coordinate fixes s, equality of the first two fixes r, and the angular interval has length less than 2π. Pullback is unitary from $L^2(\Omega)$ to $\mathcal{H} = L^2(\mathcal{Q}, r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s)$. Direct differentiation gives the form

$$q[v] = \int_{\mathcal{Q}} \Big(|v_r|^2 + r^{-2}|v_t|^2 + |Xv|^2\Big)\, r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s, \qquad X = \partial_s - \beta(s)\partial_t. \tag{2.1}$$

The Dirichlet domain is the closure of the pulled-back compact smooth tests in the norm $(q[v]+\|v\|^2)^{1/2}$. For the auxiliary radial-Neumann variant we instead take the maximal finite-energy form domain with zero trace on the two angular faces, allowing free traces at $r = a, b$. Local restrictions are H^1 because β is bounded on bounded s-intervals. These definitions also specify the mixed-boundary forms used below; no boundary condition is imposed by a formal differential expression alone.

Lemma 2.1 (Half-tail estimate)**.** *Suppose $\beta \geq B > 0$ on $s > S$. For either radial boundary convention, restrictions of form vectors satisfy*

$$\int_{s>S} |Xv|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s \geq \frac{\pi^2 B^2}{4\alpha^2} \int_{s>S} |v|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s. \tag{2.2}$$

There is an analogous estimate on the negative half-tail when its twisting speed has the same positive lower bound.

Proof. At fixed physical polar angle $\varphi = t+\theta(s)$, X becomes differentiation in s. The change from (t,s) to (φ,s) has Jacobian one, with φ interpreted modulo 2π. Since $\theta' = \beta > 0$, each connected permitted vertical-line component in the half-tail corresponds to a θ-interval of width at most α. Its length in s is at most α/B. The genuine lateral endpoints are Dirichlet. At most one component can meet the artificial cap $s = S$; that component has at least one Dirichlet endpoint and may have a free trace at the cap. The one-dimensional Dirichlet–Neumann Poincaré inequality on an interval of length L has constant $\pi^2/(4L^2)$. Apply it on each component and integrate in r, φ. The argument first applies to locally smooth vectors with angular zero trace and then passes to the form domain by approximation. The radial boundary convention does not enter the estimate. The negative tail is handled in the same way. □

Proposition 2.2. *The sector operator, with either radial boundary convention, has compact resolvent for* (1.3)*.*

Proof. On $|s| > S$, use Lemma 2.1 with $B = c(1+S^2)^{p/2}$. The mass of a form-norm bounded family in these tails tends uniformly to zero as $S \to \infty$. On a bounded s-interval, (2.1) controls the ordinary H^1 norm: v_t and v_r are controlled directly, while $v_s = Xv + \beta v_t$. The measure $r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s$ is equivalent to Lebesgue measure there. Rellich compactness on the bounded coordinate box and uniform tail tightness imply compactness of the form-domain embedding into $\mathcal{H}$. □

3 An exact frozen-box comparison

Fix ρ in a compact positive interval, a constant $B > 0$, radial length $h > 0$ and longitudinal length $L > 0$. On the rectangular box use Lebesgue measure and the constant-coefficient form

$$q_{\rho,B}[v] = \int \Big(|v_r|^2 + \rho^{-2}|v_t|^2 + |v_s - Bv_t|^2\Big)\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s. \tag{3.1}$$

Angular faces are always Dirichlet. The radial condition is either Dirichlet on both faces, denoted D, or Neumann on both, denoted N. Longitudinal faces are Dirichlet. Denote the corresponding count by $D_{\mathsf{b}}(h, L, \rho, B; E)$. Put

$$\mathcal{J}_{\mathsf{D}} = \{1, 2, \ldots\}, \qquad \mathcal{J}_{\mathsf{N}} = \{0, 1, \ldots\}, \qquad \nu_n = \frac{\pi n}{\alpha}, \tag{3.2}$$

and define

$$J_{\mathsf{b}}(h, \rho, B; E) = \frac{\sqrt{1 + B^2\rho^2}}{\pi} \sum_{\substack{j \in \mathcal{J}_{\mathsf{b}} \\ n \geq 1}} \Big[E - (\pi j/h)^2 - (B^2 + \rho^{-2})\nu_n^2\Big]_+^{1/2}. \tag{3.3}$$

All sums at a fixed energy are finite. For negative energy, all counts and positive-part sums in this section are defined to be zero.

Lemma 3.1 (Uniform longitudinal boundary comparison)**.** *For $\eta \in (0, 1)$ there is $C_\eta > 0$, independent of B, ρ, h, L, E and the radial condition, such that*

$$\frac{L}{1+\eta} J_{\mathsf{b}}\left(h, \rho, B; E - \frac{C_\eta}{L^2}\right) \leq D_{\mathsf{b}}(h, L, \rho, B; E) \leq L J_{\mathsf{b}}(h, \rho, B; E). \tag{3.4}$$

Proof. First replace the longitudinal boundary condition by $v(L) = e^{i\vartheta} v(0)$. The longitudinal frequencies are $\xi = (2\pi k + \vartheta)/L$, $k \in \mathbb{Z}$. At a fixed frequency, completion of the angular square gives

$$\rho^{-2}|w_t|^2 + |i\xi w - B w_t|^2 = (B^2 + \rho^{-2}) \left| w_t - \frac{iB\xi}{B^2 + \rho^{-2}} w \right|^2 + \frac{\xi^2}{1 + B^2\rho^2} |w|^2. \tag{3.5}$$

The constant-modulus angular gauge preserves the two angular Dirichlet conditions. Its phase is independent of r, since ρ is frozen. The exact quasiperiodic eigenvalues are therefore

$$(\pi j/h)^2 + (B^2 + \rho^{-2})\nu_n^2 + \frac{\xi^2}{1 + B^2\rho^2}. \tag{3.6}$$

The longitudinal Dirichlet form domain is contained in every quasiperiodic form domain. Its count is at most the count for each ϑ. Averaging the latter over $[0, 2\pi)$ converts the frequency lattice exactly into $L\,\mathrm{d}\xi/(2\pi)$. Integrating the intervals allowed by (3.6) proves the upper bound in (3.4).

For the lower bound set $d = L/(1+\eta)$. On a circle of length Kd, for K sufficiently large, choose K translates of a smooth square partition of unity, with spacing d, supports of length L, and

$$\sum_{k=1}^{K} \chi_k^2 = 1, \qquad \sum_{k=1}^{K} |\chi_k'|^2 \leq C_\eta L^{-2}. \tag{3.7}$$

Such a partition follows by normalizing translated nonnegative bumps which are positive on the spacing intervals and supported in their slightly enlarged intervals. The IMS identity is

$$\sum_k q_{\rho,B}[\chi_k v] = q_{\rho,B}[v] + \int \sum_k |\chi_k'|^2 |v|^2. \tag{3.8}$$

Indeed $(\partial_s - B\partial_t)(\chi_k v) = \chi_k X_B v + \chi_k' v$; the localization error has no factor depending on B.

Let $N_{\mathrm{per}}(Kd, E_0)$ be the periodic count on that circle. The isometric localization $v \mapsto (\chi_1 v, \ldots, \chi_K v)$ and min–max give

$$N_{\mathrm{per}}(Kd, E_0) \leq K D_{\mathsf{b}}(h, L, \rho, B; E_0 + C_\eta L^{-2}). \tag{3.9}$$

For completeness, if the left spectral subspace had greater dimension than the direct sum of the right spectral subspaces, some nonzero vector would localize orthogonally to every right low-energy subspace. Its summed localized energy would be at least the right threshold times its squared norm, contradicting (3.8) and the strict energy bound on the left finite-dimensional spectral subspace. At fixed E_0, h, ρ, B, d, only finitely many bands in (3.6) are active. The count for each band is a lattice count in one bounded frequency interval, with an error at most two. Consequently

$$\lim_{K\to\infty} \frac{1}{K} N_{\mathrm{per}}(Kd, E_0) = dJ_{\mathsf{b}}(h, \rho, B; E_0). \tag{3.10}$$

Divide (3.9) by K, pass to this limit, and take $E_0 = E - C_\eta L^{-2}$. This proves the lower bound. □

Lemma 3.2 (Uniform radial sum)**.** *Let ρ range in a fixed compact positive interval. Uniformly in $B > 0$, $h > 0$, $E \geq 0$ and $\mathsf{b} \in \{\mathsf{D}, \mathsf{N}\}$,*

$$J_{\mathsf{b}}(h, \rho, B; E) = \frac{h}{4\pi}\sqrt{1 + B^2\rho^2} \sum_{n\geq 1} \left[E - (B^2 + \rho^{-2})\nu_n^2\right]_+ + O(E). \tag{3.11}$$

The implied constant is independent of the growing twist B.

Proof. For $A \geq 0$, decreasing-sum comparison gives

$$\sum_{j\geq 1} \left[A - (\pi j/h)^2\right]_+^{1/2} = \frac{h}{4}A + O(\sqrt{A}). \tag{3.12}$$

The integral is $hA/4$ and the sum-integral error is bounded by $\sqrt{A}$. The inclusion of $j = 0$ changes only that error bound. Put $a_B = B^2 + \rho^{-2}$. A further decreasing-sum comparison yields

$$\sum_{n\geq 1} \left[E - a_B\nu_n^2\right]_+^{1/2} \leq \int_0^\infty \left[E - a_B(\pi x/\alpha)^2\right]_+^{1/2} \mathrm{d}x = \frac{\alpha E}{4\sqrt{a_B}}. \tag{3.13}$$

The identity $\sqrt{1 + B^2\rho^2} = \rho\sqrt{a_B}$ cancels the possible large coefficient in the summed error of (3.12). Substitution into (3.3) proves the result. □

4 The phase sum and its asymptotic

For a fixed opening α define

$$\mathcal{I}_\alpha(\Lambda) = \frac{1}{4\pi}\int_{\mathbb{R}}\int_a^b \sqrt{1 + \beta(s)^2 r^2} \sum_{n\geq 1} \left[\Lambda - (\beta(s)^2 + r^{-2})\nu_n^2\right]_+ \mathrm{d}r\,\mathrm{d}s. \tag{4.1}$$

All terms vanish if $\beta(s)\nu_1 \geq \sqrt{\Lambda}$. In particular, the integral is finite and is supported in $|s| \leq K_0\Lambda^{1/(2p)}$ for some fixed K_0 and all sufficiently large Λ.

Lemma 4.1. *For* (1.3)*,*

$$\mathcal{I}_\alpha(\Lambda) \sim C_\alpha \Lambda^{d_p}, \tag{4.2}$$

where C_α is (1.8)*. The integrand in* (4.1)*, including its $1/(4\pi)$ factor, is bounded by $C\Lambda^{3/2}$ uniformly in $r \in [a, b]$ and $s \in \mathbb{R}$.*

Proof. Write $a_s = \beta(s)^2 + r^{-2}$. There are at most $\alpha\sqrt{\Lambda}/(\pi\sqrt{a_s})$ active angular terms. Each positive part is at most Λ, while $\sqrt{1+\beta(s)^2 r^2} = r\sqrt{a_s}$. This proves the pointwise bound.

Rescale $s = \Lambda^{1/(2p)} y$. The normalized integrand for $\mathcal{I}_\alpha(\Lambda)/\Lambda^{d_p}$ converges for $y \neq 0$ to

$$\frac{r}{4\pi} c|y|^p \sum_{n\geq 1} \left[1 - c^2|y|^{2p}\nu_n^2\right]_+ . \tag{4.3}$$

At such y only finitely many limiting bands can contribute, and the positive parts are continuous even at band thresholds. The preceding pointwise bound gives a bounded majorant after normalization, including near $y = 0$, where the number of active modes can increase without bound. The rescaled support is contained in a fixed compact y-interval. Dominated convergence therefore applies. Tonelli's theorem permits termwise integration of the nonnegative limiting sum. Direct integration over both half-lines gives

$$\int_{\mathbb{R}} c|y|^p \left[1 - c^2|y|^{2p}\nu_n^2\right]_+ \mathrm{d}y = \frac{4p\, c^{-1/p}}{(p+1)(3p+1)} \nu_n^{-1-1/p}. \tag{4.4}$$

The series of these integrals converges since $1 + 1/p > 1$. Multiplying by $\int_a^b r\,\mathrm{d}r/(4\pi)$ gives precisely (1.8). □

5 Uniform localization and the sector Weyl law

We prove the sector asymptotic simultaneously for Dirichlet and Neumann radial faces. Angular faces remain Dirichlet throughout.

Lemma 5.1 (Coefficient freezing)**.** *On a radial slab of width at most h, choose ρ in that slab. On a longitudinal interval let $B = \beta(s_0)$ and $\Delta = \sup|\beta - B|$. If $h + \Delta$ is small, the Rayleigh quotients of* (2.1) *and* (3.1) *satisfy*

$$(1 - C(h+\Delta))\frac{q_{\rho,B}[v]}{\|v\|_{L^2}^2} \leq \frac{q[v]}{\|v\|^2_{L^2(r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s)}} \leq (1 + C(h+\Delta))\frac{q_{\rho,B}[v]}{\|v\|_{L^2}^2}. \tag{5.1}$$

The constants are uniform in B and in the radial Dirichlet or Neumann conditions, with r, ρ in a fixed compact positive interval.

Proof. The density and r^{-2} differ from their frozen values by relative $O(h)$. For the mixed term write $Xv = X_B v - (\beta - B)v_t$. Young's inequality with parameter Δ when $\Delta > 0$ bounds its change by $C\Delta(|X_B v|^2 + |v_t|^2)$ for $\Delta \leq 1$. The angular part $\rho^{-2}|v_t|^2$ absorbs the latter term. For $\Delta = 0$ there is no longitudinal coefficient change. Finally compare the two squared norms by their density factors. No differentiation of the density or of a varying-radius gauge is involved. □

Proposition 5.2 (Sector asymptotic)**.** *For $\alpha \in (0, 2\pi)$,* (1.6) *holds with coefficient C_α for the sector, for either radial boundary convention.*

Proof. Fix $\eta \in (0,1)$ until the last step and choose

$$0 < \varepsilon < \min\left\{\frac{1}{4}, \frac{1}{4p}\right\}, \qquad h = \Lambda^{-1/4}, \qquad \ell = \Lambda^{-1/2+\varepsilon}, \qquad S = K\Lambda^{1/(2p)}. \tag{5.2}$$

The constant K is fixed and large enough for the tail estimates below. For $|s| \leq S + 1$ the absolute oscillation of β on intervals of length $O_\eta(\ell)$ is bounded by

$$\Delta_\Lambda \leq C_\eta \ell (1+S)^{\max\{p-1,0\}} = o(1). \tag{5.3}$$

For $p > 1$ the exponent in this bound is $\varepsilon - 1/(2p) < 0$; for $p \le 1$ it is $-1/2 + \varepsilon < 0$. Thus freezing is absolute, not merely relative to the possibly large value of β.

Upper bound. Subdivide (a, b) into radial intervals of length h, apart from at most one shorter final interval, and impose Neumann conditions at their interfaces. This enlarges the form domain; there is no mixed radial derivative in (2.1). For an original Dirichlet radial face it is also permissible to relax that face to Neumann in this upper comparison.

On the entire s-axis take a smooth square partition with spacing ℓ, supports of length $L = (1 + \eta)\ell$, and squared derivative sum at most $C_\eta \ell^{-2}$, constructed as in (3.7). The exact IMS error is at most $C_\eta \ell^{-2} \|v\|^2 = o(\Lambda)\|v\|^2$. Each localized vector has Dirichlet longitudinal endpoints and retains the angular zero traces. If a support lies entirely in $|s| > S$, Lemma 2.1 puts its form above 2Λ by choosing K large enough. The same is true in every radial slab. Only $O_\eta(S/\ell)$ longitudinal supports can therefore contribute at the shifted energy. There is no Neumann cut in the longitudinal direction, and no assumption of confinement in arbitrarily short Neumann boxes.

On each contributing box freeze r and β. By (5.3), Lemma 5.1 and the IMS bound, there exists $\tau_\Lambda \to 0$, uniform over these boxes, so that min–max bounds the original count by the sum of the frozen radial-Neumann, longitudinal-Dirichlet counts at $E_+ = (1 + \tau_\Lambda)\Lambda$. The localization map into the direct sum is isometric; the spectral-subspace dimension argument used in (3.9) applies also here. All remaining summands are zero by the tail estimate.

Apply the upper part of Lemma 3.1 and Lemma 3.2. The total remainder is bounded by

$$C_\eta \frac{S}{h\ell}\, \ell\Lambda = O_\eta(S\Lambda/h) = O_\eta(\Lambda^{d_p - 1/4}) = o(\Lambda^{d_p}). \tag{5.4}$$

The leading box sum has radial cell lengths $h_j \le h$ and longitudinal support lengths $(1+\eta)\ell$. Its spacing cells in s have length ℓ. Consequently its geometric support-length factor relative to the corresponding phase integral is $1 + \eta$.

Here is a uniform way to justify the phase-sum comparison, including all thresholds. On an active box put $a(s, r) = \beta(s)^2 + r^{-2}$ and $w(s, r) = \sqrt{1 + \beta(s)^2 r^2}$. Since $\beta \ge c$, both a and w vary relatively by $O(h + \Delta_\Lambda)$ from their frozen values. For $\gamma \to 0$ uniform on the boxes, the inequalities $a_0 \le (1 + \gamma)a$ and $a_0 \ge (1 - \gamma)a$ squeeze $[E - a_0\nu_n^2]_+$ between the corresponding positive parts at energies $E/(1 \pm \gamma)$, multiplied by $1 \pm \gamma$. Multiplying by the analogous bounds for w preserves the sum over *every* n. Thus the frozen principal sums are bounded above by $(1 + \eta)(1 + o(1))\mathcal{I}_\alpha((1 + o(1))\Lambda)$. The chosen K contains the support of each such integral. Grid cells at the ends of this active range are harmless for the same comparison, or contribute at most $O(\ell\Lambda^{3/2})$ by Lemma 4.1. We conclude that

$$\limsup_{\Lambda\to\infty} \frac{N(\Lambda)}{\Lambda^{d_p}} \le (1 + \eta)C_\alpha. \tag{5.5}$$

Lower bound. Insert Dirichlet boundaries on disjoint radial intervals of length h and longitudinal intervals of length ℓ inside $(-S, S)$. This restricts the original form domain, for either original radial boundary convention. Freeze the coefficients as above. The lower part of Lemma 3.1 costs $C_\eta \ell^{-2} = o(\Lambda)$ and contributes the factor $1/(1 + \eta)$. The radial sum remainder is again (5.4). The same positive-part comparison now gives a lower phase integral at energy $(1 - o(1))\Lambda$. Discarded radial endpoint layers have total phase contribution at most $O(hS\Lambda^{3/2})$, and discarded longitudinal endpoint layers at most $O(\ell\Lambda^{3/2})$. Both are $o(\Lambda^{d_p})$. Therefore

$$\liminf_{\Lambda\to\infty} \frac{N(\Lambda)}{\Lambda^{d_p}} \ge \frac{C_\alpha}{1 + \eta}. \tag{5.6}$$

Use Lemma 4.1 in both bounds and let $\eta \downarrow 0$ after $\Lambda \to \infty$. This proves the proposition. □

6 Curvilinear polar strips

We now prove Theorem 1.1. This step uses fixed radial subdivision before taking a high-energy limit; it requires no uniform asymptotic for a moving angular boundary.

Proof of Theorem 1.1. The coordinate map is injective under (1.1), just as in the sector case. Its form is (2.1) on $a < r < b$, $l(r) < t < u(r)$. The domain has angular Dirichlet traces on these two C^1 graphs. At a fixed r, the half-tail slicing argument uses interval width $\alpha(r) \le \alpha_{\max} < 2\pi$. Lemma 2.1 with $\alpha_{\max}$ and local Rellich compactness on the bounded Lipschitz coordinate domains prove compact resolvent.

Fix a finite partition $a = r_0 < r_1 < \cdots < r_M = b$. On $I_j = [r_{j-1}, r_j]$ put

$$\alpha_j^- = \min_{I_j} u - \max_{I_j} l, \qquad \alpha_j^+ = \max_{I_j} u - \min_{I_j} l. \tag{6.1}$$

Uniform continuity and the positive minimum of α imply that, for a sufficiently fine mesh, $\alpha_j^- > 0$ and $\alpha_j^+ < 2\pi$. The intervals

$$(\max_{I_j} l, \min_{I_j} u), \qquad (\min_{I_j} l, \max_{I_j} u) \tag{6.2}$$

define the inner and outer sectors on that radial cell.

For the lower comparison take the disjoint inner sector tubes, imposing Dirichlet conditions on their added boundaries. They are genuine subdomains of Ω, so their direct-sum count is at most $N_\Omega(\Lambda)$.

For the upper comparison first impose only Neumann interfaces at the radial partition. On each resulting slab extend a form vector by zero across its angular Dirichlet walls into the outer sector in (6.2); retain free radial traces at the artificial radial faces. This extension is H^1 on every bounded longitudinal piece. Indeed the weak derivative of the angular zero extension has no boundary distribution: every possible wall term, including the terms involving $l'(r)$ or $u'(r)$ in its radial derivative, is multiplied by the zero angular trace. Equivalently one may use local approximation by vectors vanishing near these Dirichlet walls. There is no extension across a radial face and hence no jump caused by its free trace. The extension preserves the norm and all terms of (2.1). Its image lies in the outer sector form domain with Neumann radial faces and Dirichlet angular faces. Domain inclusion and min–max give an upper bound by the direct sum of those outer sector counts.

For each fixed mesh there are finitely many sectors. Apply Proposition 5.2 to each one; shifting its angular interval by a constant has no effect on its form. With

$$K_{p,c} = \frac{p\,\zeta(1+1/p)c^{-1/p}}{\pi^{2+1/p}(p+1)(3p+1)}, \tag{6.3}$$

the resulting inequalities are

$$\begin{aligned} K_{p,c} \sum_j (\alpha_j^-)^{1+1/p} \int_{I_j} r\,\mathrm{d}r &\le \liminf_{\Lambda\to\infty} \frac{N_\Omega(\Lambda)}{\Lambda^{d_p}}, \\ \limsup_{\Lambda\to\infty} \frac{N_\Omega(\Lambda)}{\Lambda^{d_p}} &\le K_{p,c} \sum_j (\alpha_j^+)^{1+1/p} \int_{I_j} r\,\mathrm{d}r. \end{aligned} \tag{6.4}$$

Refine the mesh only after these high-energy limits. Both sums converge to $\int_a^b r\alpha(r)^{1+1/p}\,\mathrm{d}r$ by uniform continuity. This proves (1.6) and (1.7). □

7 Consequences and scope

The angular factor in (1.7) distinguishes the law from an area-only expression: $|\omega| = \int_a^b r\alpha(r)\,\mathrm{d}r$, whereas the counting coefficient involves $\alpha^{1+1/p}$. The angular centerline $(l+u)/2$ and the derivatives of the endpoints do not enter the leading term. This is a consequence of the geometric squeeze.

Two standard spectral consequences can be read directly from the result. They are included to specify the normalization of the counting law.

Corollary 7.1. *Write the eigenvalues in nondecreasing order with multiplicity as λ_k, and put $d = d_p$ and $C = C_\omega$. Then*

$$\lambda_k \sim (k/C)^{1/d}, \qquad k \to \infty, \tag{7.1}$$

and

$$\mathrm{Tr}(e^{-tH_\Omega}) \sim C\Gamma(d+1)t^{-d}, \qquad t \downarrow 0. \tag{7.2}$$

Proof. For every $\epsilon > 0$, the two bounds $(C-\epsilon)\Lambda^d \le N(\Lambda) \le (C+\epsilon)\Lambda^d$ hold for all sufficiently large Λ. Inverting these bounds, with arbitrarily small strict threshold shifts if necessary, gives (7.1). The counting asymptotic also gives a global bound $N(\Lambda) \le C_0(1+\Lambda^d)$ for $\Lambda \ge 0$. Integration against the spectral counting measure yields

$$\mathrm{Tr}(e^{-tH_\Omega}) = t\int_0^\infty e^{-t\Lambda}N(\Lambda)\,\mathrm{d}\Lambda.$$

Changing variables $x = t\Lambda$, multiplying by t^d and applying dominated convergence proves (7.2). In particular the trace is finite for every $t > 0$. □

The theorem concerns each fixed $p > 0$. The specified smooth model speed provides both the power scaling and the absolute-oscillation control in (5.3). A function merely asymptotic to $c|s|^p$ need not satisfy that derivative control. Likewise, cross-sections whose angular width pinches, which meet the rotation axis, or which have multiple angular components require additional approximation arguments. The present result therefore gives an exact law in a concrete family within the question of [5, 6], rather than a general resolution of that question.

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